THE EVOLUTION OF COOPERATION IN A NON-COOPERATIVE GAME WITH PUBLIC GOODS

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I. Introduction

There is a common tendency in economics and related disciplines to presume that government intervention is warranted in the provision of collective consumption goods and the management of common property resources.

In the tradition of Aristotle (Stroup, 1991), Hume (Olsen, 1965) and Hardin (1968), individuals following their inevitably myopic self-interest will underprovide public goods and overexploit common property resources.

The presumption that public goods will be underprovided can be formally demonstrated as a Nash equilibrium in a non-cooperative game. In this model, each player chooses to provide that quantity of good which will maximize his own utility, assuming that the amount provided by other players will remain the same as in the previous period. In equilibrium each of the n players equates his marginal rate of substitution to the full cost of a unit of the public good. Therefore, the sum of the marginal rates of substitution is n times the marginal rate of transformation, i.e. the public good is underprovided and the degree of underprovision increases with the number of potential

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cooperators (Roberts and Holdren, 1972; Cornwall, 1984).

With more sophisticated strategies, however, it is plausible that cooperation can be sustained by punitive strategies directed at defecting players. Indeed, any target solution that becomes the focal point of such punitive action and which is superior to the Nash equilibrium can be so sustained (Aumann, 1981). This "folk Theorem" has two severe limitations. First it applies equally to efficient and inefficient outcomes. Second, it provides no guidance on how a particular focal point becomes prominent.

In this paper, we derive conditions under which public goods could be under, over, or efficiently provided, even in a non-cooperative setting. In Section II of this paper, we show that efficiency in the voluntary provision of public goods can be achieved if consumers underestimate the quantity of public goods to be provided by others. In particular, we show that efficiency can occur under various combinations of numbers of players (n), elasticities of substitution (a), and the degrees of underestimation of public good provision by others, with the degree of underestimation being measured by 0 which is the proportional underestimation of the actual public good provision by others.

In Section III, we discuss how an efficient solution, once achieved, has a natural prominence and may be sustained as a spontaneous order (Hayek, 1960). This provides a fundamental theory of government as the constitution of cooperation in an economy with public goods and thus complements Nozick's (1974) minimal state. Section IV provides a brief summary, while Appendix A provides derivations of equations (1) through (5).

II. A Model of the Voluntary Provision of a Public Good

In contrast to the Nash-Cournot model, wherein players assume that other players' provisions of the public good will remain the same as in the prior period, and equilibrium occurs when that assumption is realized by all n players, we provide a more general analysis wherein assumptions as to others' provisions may be based on other than the prior period's behavior and wherein those assumptions may not be realized. We begin by presenting the following two-player game model which allows for explicit treatment of conjectures regarding the other player's strategy.

In this model both players' preferences are representable by Cobb-Douglas utility functions. Specifically, suppose that two individuals, Ken and Ben, maximize their utility functions:

$$U_K = X_K^{\alpha} (S_K + S_B^{a})$$

$$U_B = X_{B} (S_B + S_k^{a})$$

subject to the budget constraints:

$$Y_K \quad P_X X_K + P_S S_K$$

$$Y_B \quad P_X X_B + P_S S_B$$

where, e.g. X_K = the amount of private good purchased by Ken

S_K = the amount of the public good provided by Ken

P_S = the (supply) price of the public good

S_B^{a} = the amount of the public good that Ken assumes Ben will provide

$S_K + S_B$ = the actual amount of public good provided.

Now suppose that $S_K^a = S_B^a = 0$, for example, because Ken and Ben are extremely risk-averse, or because neither player has provided the public good in the previous period. In this case, the sum of the quantities of the public good provided by the players will be efficient.[2] An intuitive explanation of this result rests on the presence of opposing forces in the model. To the extent that each individual's provision of the public good generates external benefits which are not taken into account in his decision calculus, there is a tendency towards underprovision. On the other hand, to the extent that individuals underestimate actual provision by the other party, e.g. $S_B^a = 0 < S_B$, there is a force towards overprovision. The latter force is greater the lower the value placed on unanticipated spillins, i.e. the lower the elasticity of substitution between the public and private good. Where the elasticity of substitution is one, i.e. the Cobb-Douglas case, and where $S_B^a = S_K^a = 0$, these two forces exactly offset one another and efficient provision results.

Although it can be shown that efficiency occurs when the elasticity of substitution is one and players assume zero provision of the public good by others, these conditions are not necessary for efficiency. In fact, we shall now generalize the model and show that efficiency can occur under various

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2. This statement is proven later in the paper in that when $\sigma = 1$ and $\phi = 0$, equations (4) or (5) hold for any n.

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combinations of n, , and .

We begin this analysis by recognizing that there are intermediate cases between that of a Nash-Cournot equilibrium (wherein assumed and realized spillins of public good provision are equal) and the zero assumed spillin case. Under the simplifying assumption that all players have identical 0's, we can derive the relationships between 0 and the level of provision of the public good relative to the two polar cases. This intermediate solution is given by:

$$\frac{S}{S^*} = \frac{1}{1 + (1-\phi)R(n-1)} \qquad (1)$$

and

$$\frac{S}{S_{NC}} = \frac{1+R(n-1)}{1 + (1-\phi)R(n-1)} \qquad (2)$$

where:

R = the negative of the partial derivative of a player's provision of the public good with respect to another player's provision of the public good, with all players assumed to have identical Rs,

S* = the level of public good provision if all players assume zero provision by others, i.e. = 1,

S_{NC} = the level of public good provision in Nash-Cournot equilibrium, i.e. = 0, and

S = the level of public good provision.

Equations (1) and (2) are derived in Appendix A.

Given the above analysis, it can be seen that 0 affects the level of public goods provision. Underestimation of others' provision of public goods distorts the ratios of private goods to public goods. Given the assumptions of identical homothetic preferences (the CES utility function being an example) and equal incomes for all n players (consumers), Equation (3) can be derived which shows how player i's ratio X_i/S is related to , n, and the ratio which player i assumes would occur if there was zero public good provision by others:

$$\frac{X_i}{S} = \frac{1+(1-\phi)(n-1)}{n} \frac{X_i^*}{S_i^*} \qquad (3)$$

where:

X_i^* = the level of private good purchased by player i if i assumes zero public goods provision by others (i.e. =1),

X_i = the level of private good purchased by player i,

S_i^* = the level of public goods purchased by player i if $\phi = 1$, and

the other variables are as previously defined.

Equation (3) is derived in Appendix A. Note that in equation (3), since [1 + (1-)(n-1)]/n < 1, even though a player's assuming some provision of public goods by others increases a player's ratio of private good to public good purchases, it reduces the ratio relative to what players assumed would occur had public good provision by others been zero. The significance of this distortion becomes apparent in the following paragraphs.

By distorting the ratio X_i/S, the marginal rate of substitution between X_i and S is altered. This, then, has effects on the efficiency of the provision of public goods. Recall that in Nash-Cournot equilibrium, there is an underprovision of the public goods since each player sets his

=1, and

marginal rate of substitution (MRS) equal to the marginal rate of transformation (MRT), resulting in the sum of the MRSs exceeding the MRT. In the Cobb-Douglas case, with the players having a unitary elasticity of substitution between public goods and private goods and assuming zero provision of the public good by others, efficiency is achieved because the "unexpected" provision of public goods by others reduces each player's MRS so that the sum of the MRSs equals the MRT. In the intermediate solution, which is between the Nash-Cournot equilibrium and the Cobb-Douglas zero assumed spillin case, there is some "unexpected" provision of public goods by others but since some provision by others was anticipated (i.e. < 1), the MRSs are not reduced as much as in the Cobb-Douglas zero assumed spillin case. To achieve efficiency, then, another factor must come into play. That factor is the elasticity of substitution between the public good and the private good. The less the substitutability between public and private goods (i.e. the smaller the), the more the MRS will decline for any given level of "unexpected" provision of public goods by others. Thus, there should, and in fact do, exist combinations of 0 (which affects the level of unexpected provision of public goods by others) and *a* which result in efficiency (the sum of the MRS equaling the MRT). Given the assumptions of n consumers with identical CES utility functions and incomes and each consumer underestimating others' provision of the public goods by the same fraction (given by 0) we can derive equations (4) and (5) which are equivalent expressions for efficient combinations of n, , and :

$$\phi = 1 - \frac{1}{n-1}\left[n\left(\frac{1}{n}\right)^{\sigma} - 1 \right] \tag{4}$$

$$\sigma = 1 - \frac{\ln\left[1+(1-\phi)(n-1)\right]}{\ln n} \tag{5}$$

Equations (4) and (5) are derived in Appendix A. Using these equations we can determine combinations of a and 0 resulting in efficiency for different numbers of players. In Figure 1 are shown combinations of a and 0 resulting in efficiency in the n = 2 and n = 100 cases.

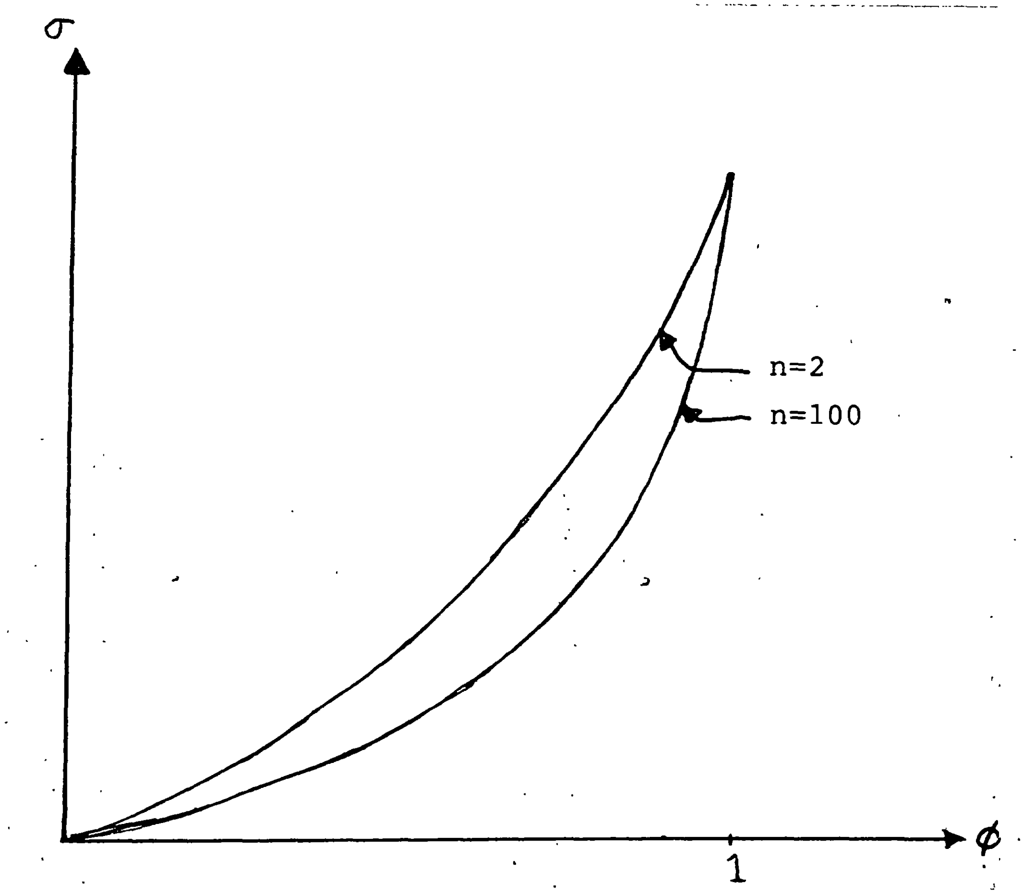

Figure 1: Combinations of a and 0 Resulting in Efficiency in Public Goods Provision For the n=2 and n=100 Cases

In Figure 1, for a given n, any combination of *a* and 0 to the southeast of the locus of efficient combinations would result in an over-provision of the public good and any combination to the northwest would result in an underprovision. Thus, under-, over- or optimal provision are possible, depending upon the combination of n, a, and 0. It is apparent from this analysis that the greater the number of players, the greater the tendency toward underprovision. This result, which supports the Hume theorem, can be readily seen in Figure 1 in that the locus of efficient combinations collapses toward the southeast as n increases.

Despite using some highly simplifying assumptions in the above analysis such as identical reaction curves and utility functions for all players, several key points have been made. The most basic point is that underprovision is not the only possible outcome when public goods are privately provided. If, for whatever reason, consumers, either totally ignore the provision of public goods by others as in the Cobb-Douglas case or partially underestimate the provision by others as in the intermediate case, efficiency may occur and we cannot presume that underprovision necessarily results. Also of significance are the more specific results of the model presented herein. Specifically, it was found, in support of the Hume theorem, that the larger the number of players the greater the tendency toward underprovision. It was also found that this tendency toward underprovision which 'arises from free riding was offset, partially or totally, by underestimation of others' provision of public goods. The greater the underestimation (i.e. the larger the), the greater the offsetting force toward overprovision, with the offsetting force being greater the less the substitutability between public and private goods (the smaller the). Further results could undoubtedly be developed, e.g. one could analyze the effects of different players having different elasticities of substitution, but the above analysis seems to capture the significant forces toward underprovision and overprovision, and the fact that these forces can result in under-, over- or optimal provision of a public good in a non-cooperative setting.

III. The Sustainability of Efficiency and the Folk Theorem

If optimal provision of voluntarily provided public goods was only sustainable if consumers continue to underestimate the provision of public goods by others, the results of the above section would be of limited interest since continued, underestimation seems unlikely over a number of decision periods. What makes the possibility of optimality in a single round setting more interesting is that once optimality is achieved, it may be sustainable over time based on the "folk theorem" rather than on the continued underestimation of others' public goods provision.

The following example will clarify the applicability of the folk theorem to our analysis. Suppose Ben and Ken have the following utility functions and budget constraints, respectively:

$$U_B = X_B(S_B + S_K) \text{ subject to } 120 \quad X_B + S_B$$

$$U_K = X_K(S_K + S_B) \text{ subject to } 120 \quad X_K + S_K$$

Assume further that both Ben and Ken, due to initial pessimism or risk aversion, assume that the other will be providing none of the public good. The payoff matrix below (Figure 2) depicts some of the possible outcomes to this game.

	$S_K=60$	$S_K=40$	$S_K=30$	$S_K=0$
$S_B=60$	4 $U_B=7200$ $U_K=7200$	$U_B=6000$ $U_K=8000$	$U_B=5400$ $U_K=8100$	2 $U_B=3600$ $U_K=7200$
$S_B=40$	$U_B=8000$ $U_K=6000$	6 $U_B=6400$ $U_K=6400$	$U_B=5600$ $U_K=6300$	$U_B=3200$ $U_K=4800$
$S_B=30$	$U_B=8100$ $U_K=5400$	$U_B=6300$ $U_K=5600$	5 $U_B=5400$ $U_K=5400$	$U_B=2700$ $U_K=3600$
$S_B=0$	3 $U_B=7200$ $U_K=3600$	$U_B=4800$ $U_K=3200$	$U_B=3600$ $U_K=2700$	1 $U_B=0$ $U_K=0$

Figure 2: Payoff Matrix Example

Given their assumptions of zero public good provision by the other, both Ben and Ken would choose to provide 60 units of the public good. That is, each, in attempting to escape from cell 1 to cells 2 and 3 respectively would provide $S_B = S_K = 60$. Thus, they would end up in cell 4, which happens to be Pareto optimal and a Lindahl equilibrium. If the two players followed Nash-Cournot behavior, they would each assume that in the next round the other player would again provide 60 units of the public good. Therefore, in round 2, each would provide 30 units of the public good and the players would end up in cell 5. If the Cournot behavior continued over a number of rounds, the outcome would gravitate toward cell 6 which would be the Nash-Cournot equilibrium.

Several points can be made from the above example. First, the outcome in cell 4 is Pareto optimal, which is consistent with our conclusion that when $= 1$ and $=0$, efficiency is achieved. Second, the Nash-Cournot equilibrium in cell 6 is suboptimal, with cell 4 being Pareto superior to it. Third, cell 4 is sustainable if Ben and/or Ken recognize that cell's desirability and threaten to match or exceed any cutback in public good provision by the other. Such a threat, in a repeated game setting with a large number of rounds, would be ample incentive for players to remain in cell 4.[3] A number of features of the cell 4 solution, which is a Lindahl equilibrium, combine to make the solution prominent in the sense of Schelling (1960). First, any change in the total contribution to the public good will decrease the economic surplus. Thus there is no possibility of a win-win strategy that departs from the Lindahl equilibrium. Moreover, the Lindahl equilibrium is "just" in the sense that all participants pay in accordance with their marginal benefits and reap the surplus due to the higher valued intramarginal units

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3. Although the above example is based on the special case of both players having identical Cobb-Douglas utility functions and zero assumed public good provision by others, alternative examples based on different numbers of players, elasticities of substitution, and degrees of underestimation could be constructed which would lead to the same demonstration that once achieved, an efficient level of the public good provision may be sustained by a threat mechanism among players.

(Lindahl, 1919). Finally, in the special case of similar tastes and incomes, the economic surplus of all participants is roughly equal.

The punitive strategies adopted constitute a governance structure of cooperation. The commitment to retaliation becomes more credible if it is encoded in the cultural norms of the group and reinforced by social sanctions such as ostracism. Enforcement against recalcitrant and thick-skinned deviants (whose presence threatens an unraveling of cooperation through envy and imitation) can be further secured by legal institutions that commit the group to punish opportunism. In this way, the spontaneous order matures into government.

IV. Summary

In this paper we have shown that, contrary to the conventional wisdom, the voluntary provision of public goods need not result in underprovision. If players underestimate the public good provision by others, then under-, over-, or optimal provision is possible, depending upon the number of players, elasticities of substitution, and the degree of underestimation. Furthermore, if optimal provision is achieved, retaliatory strategies can be used to sustain the efficient solution as a spontaneous order.

APPENDIX A

Derivation of Equations (1) - (5)

In deriving equations (1) and (2) we use the following definitions:

S_i = the level of public good provision by player i,

S_i^* = the level of public good provision by player i, if =1

and the other variables are as previously defined. Given the assumption that all n players have identical reaction curves, player i's reaction curve would be given by:

$$S_i = S_i^* - (1-\phi)R\sum_{j\neq i} S_j$$

and since $S_i = S_j$ for all i and j we can substitute $(n-1)S_i$ for $\sum_{j\neq i} S_j$ to get:

$$S_i = S_i^* - (1-\phi)R(n-1)S_i$$

which can be rearranged as:

$$S_i = \frac{S_i^*}{1 + (1-\phi)R(n-1)}$$

and

$$\frac{S_i}{S_i^*} = \frac{1}{1 + (1-\phi)R(n-1)}$$

Since all players behave identically and would provide identical levels of the public good, $S = nS_i$ and $S^* = nS_i^*$. Thus we get equation (1):

$$\frac{S}{S^*} = \frac{1}{1 + (1-\phi)R(n-1)} \qquad (1)$$

Note that in the Nash-Cournot equilibrium, = 0, so

$$S_{NC} = \frac{nS_i^*}{1 + R(n-1)}$$

which when divided into

$$S = \frac{nS_i^*}{1 + (1-\phi)R(n-1)}$$

yields equation (2):

$$\frac{S}{S_{NC}} = \frac{1 + R(n-1)}{1 + (1-\phi)R(n-1)} \qquad (2)$$

In deriving equation (3) we assume that there are n identical consumers with homothetic preferences for X and S. Each player i is subject to the budget constraint $Y_i \quad X_i + kS_i$, where the private good X is the numeraire and k is the marginal rate of transformation. We shall use Figure A1 to depict the intermediate solution wherein each player underestimates by fraction Θ the provision of public good by others:

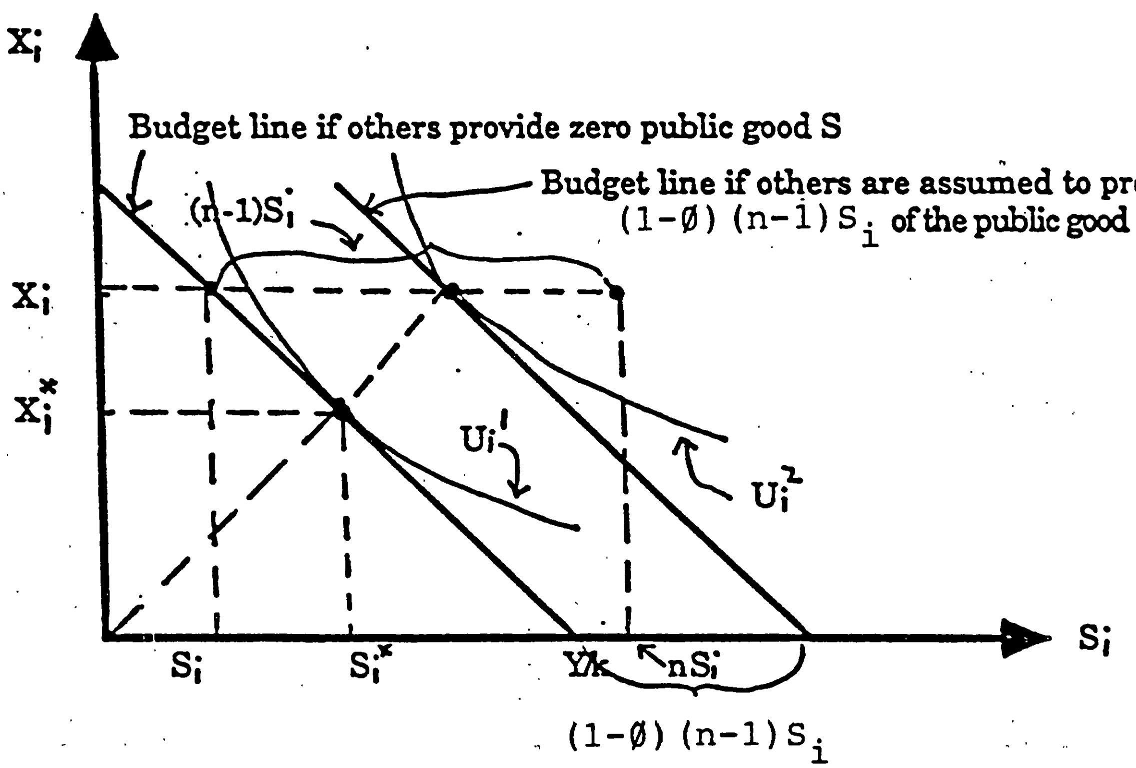

Figure A1: Solution with n Identical Players

Given the assumption of zero provision of public good S by others, player i would select point (S_i*, X_i*). If, instead, player i assumes that others will be providing some of the public good and that all players other than i are identical then the player will assume that provision by others will total $(1-\ )(n-1)S_j$ (where j i). Given our assumptions of identical consumers with identical 's, all players provide the same level of public good so $S_j = S_i$ and we can specify that the solution is where

$$S_i = S_i^*\left[1 + \frac{(1-\phi)(n-1)S_i}{Y/k}\right] - (1-\phi)(n-1)S_i \qquad \text{(A1)}$$

This expression states that the solution occurs where each consumer's purchase of S equals the consumer's chosen consumption of S less the assumed provision by others. Equation (A1) can be manipulated to yield:

$$S_i = \frac{S_i^*}{1 + (1-\phi)(n-1) - \frac{(1-\phi)(n-1)S_i^*}{Y/k}} \qquad \text{(A2)}$$

An expression for X_i can be derived from

$$X_i = X_i^*\left[1 + \frac{(1-\phi)(n-1)S_i}{Y/k}\right]$$

yielding:

$$X_i = X_i^*\left[1 + \frac{(1-\phi)(n-1)S_i^*\ k/Y}{1 + (1-\phi)(n-1) - (1-\phi)(n-1)S_i^*\ k/Y}\right]$$

Taking the ratio of X_i to S_i we get, after some simplification:

$$\frac{X_i}{S_i} = \left[1 + (1-\phi)(n-1)\right]\frac{X_i^*}{S_i^*}$$

This expression shows that player i's ratio of private to public good purchases decreases when the player underestimates the provision of the public good by others. Note too that X_i/S_i is an increasing function of n.

To complete our derivation of equation (3), we now note that player i's actual consumption of S equals nS_i since all n players behave identically. Thus we get:

$$\frac{X_i}{S} = \frac{1 + (1-\phi)(n-1)}{n} \frac{X_i^*}{S_i^*} \qquad (3)$$

To derive equations (4) and (5) we make use of the efficiency condition that MRS equal MRT for a public good. It can be shown for a CES utility function as follows:

$$U_i = \left[\alpha_i X_i^{-\beta_i} + (1 - \alpha_i) S^{-\beta_i}\right]^{-1/\beta_i}$$

where ${}_i$ = $1/({}_i + 1)$, and assuming all n players are identical, after dropping subscripts to and that:

$$\Sigma_i MRS = n \left(\frac{1 - \alpha}{\alpha}\right)\left(\frac{X_i}{S}\right)^{1/\sigma}$$

This expression we set equal to k (the MRT) to find an expression for efficient outcomes and substitute equation (3) into the X_i/S term, yielding:

$$n \left(\frac{1 - \alpha}{\alpha}\right)\left[\left(\frac{1 + (1-\phi)(n-1)}{n}\right)\left(\frac{X_i^*}{S_i^*}\right)\right]^{1/\sigma} = k$$

which can be rearranged as:

$$n \left(\frac{1 - \alpha}{\alpha}\right)\left(\frac{X_i^*}{S_i^*}\right)^{1/\sigma}\left(\frac{1 + (1-\phi)(n-1)}{n}\right)^{1/\sigma} = k \qquad (A4)$$

Noting that if player i assumes zero provision by others then the player sets

$$\left(\frac{1 - \alpha}{\alpha}\right)\left(\frac{X_i^*}{S_i^*}\right)^{1/\sigma} = k$$

we can substitute k for its equivalent expression on the left side of equation (A4) and then divide both sides by nk yielding:

$$\left[\frac{1 + (1-\phi)(n-1)}{n}\right]^{1/\sigma} = \frac{1}{n} \qquad \text{(A5)}$$

as an expression for efficient combinations of 0, a, and n. Equations (4) and (5) are merely rearrangements of equation (A5) putting 0 and a on the left sides of the equations.

REFERENCES

Aumann, R. (1981). "Survey of Repeated Games," in Aumann, et al. (eds.), Essays in Game Theory, Mannheim: Bibliographisches Institut.

Cornwall, R. (1984). Introduction to the Use of General Equilibrium Analysis, New York: North Holland Publishing Co.

Hardin, G. (1968). "The Tragedy of the Commons," Science, 162: 1243-1248.

Hayek, F. (1960). The Constitution of Liberty, London: Routledge and Kegan Paul.

Lindahl, E. (1962). "Just Taxation—A Positive Solution", in R. Musgrave and A. Peacock, Classics in the Theory of Public Finance, New York: MacMillan, 168-176.

Nozick, R. (1974). Anarchy, State and Utopia. New York: Basic Books.

Olson, M. (1965). The Logic of Collective Action, Cambridge, MA: Harvard University Press.

Roberts & Holdren. (1972). Theory of Social Process, Ames, Iowa: Iowa State Unviersity Press.

Schelling, T. (1960). The Strategy of Conflict, Cambridge: Harvard University Press.

Stroup, R. (1991). "Controlling Earth's Resources: Markets or Socialism?" Population and Environment, 12(3): 265-284.