"Public Ownership: Three Proposals for Resource Allocation"

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ABSTRACT

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While there is a quite clear picture of the rights that private ownership bestows upon the owner, it is not clear what property rights the public have by virtue of their owning a thing collectively. We ask: how should a planner, whose job is to respect public ownership of some productive assets, in conjunction with private ownership of some inputs, allocate resources?

We insist throughout on the desideratum that: (1) the final allocation be Pareto efficient. We propose three additional desiderata: (2) equal division of benefits derived from public ownership; (3) equal returns to the use of privately owned inputs; (4) universal gain from improvements in the publicly owned asset. No more than one of (2). (3) and (4) is in general compatible with (1). Each of the three compatible pairs of desiderata characterizes a proposal for public ownership. We call the equal benefit solution the one characterized by (1)-(2). the proportional solution the one characterized by (1)-(3) and the constant returns equivalent mechanism the one characterized by (1)-(4). A discussion of these ideas in different institutions (a publicly owned firm, a cooperative and a common pool resource) leads us to advocate the proportional solution.

Our main formal results are (a) the existence of proportional solutions in convex economies with arbitrary consumption sets and many inputs, outputs and firms and (B) the axiomatic characterization of the constant returns equivalent mechanism. Some simulations illustrate a surprising similarity between the proportional solution and the constant returns equivalent mechanism.

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requires, first, having a conception of what the allocation of resources should be in an economy where some assets are publicly owned and some (such as labor) are privately owned. Socialist theory has not grappled with this question. To wit, suppose an economic environment is specified, that is, the preferences of the agents, the technology, and the private and public endowments of the agents are delineated. What resource allocation would an economist predict, or recommend, as consistent with these data? If all endowments are private, and the various convexity assumptions, etc., hold, the prediction or recommendation is the Walrasian equilibrium. We have no such recommendation when some of the endowments are stipulated to be public, that is. to belong to everyone.

Walrasian equilibrium is the paradigmatic prediction for a private ownership economy because it is viewed as the outcome of in economy in which the institution of competitive markets is well established. And we have historical evidence to indicate that the institution of the market evolves in an economic environment in which endowments are privately owned. We have very little historical experience, however, with public ownership; there is no such clear institutional partner to the property rights of public ownership. In particular, we believe the historical experience of central allocation is too short to constitute the definitive choice for such a partner. In this paper, we take a normative approach to the problem of defining the economic consequences of public ownership. What resource allocation mechanism might one recommend as respecting the private and public property rights that agents in such a mixed economy possess?

Sections 2-5 study the problem in a simple economy, with one publicly owned technology, requiring one private input (labor), and producing one output. Section 6 reports briefly on some simulation results, and Section 7 shows how our proposals generalize to economies with many inputs, outputs, and industries.

1. INTRODUCTION

Even in capitalist economies, it is seldom the case that all productive assets are privately owned. And in socialist economies, in which most productive assets are publicly owned, some factors remain privately owned, in particular, labor. While there is quite a clear picture of the rights that private ownership bestows upon the owner, it is not clear what property rights the public have by virtue of their owning a thing collectively. Public ownership is procedurally defined in roughly the same way as private ownership: the public has the right to 'use and abuse' its collective assets as it sees fit. Because the public is not of one mind, this definition is not very helpful. Indeed, various impossibility theorems might be interpreted to suggest that there almost never exist interesting economies in which public ownership is meaningful.

We shall not delve into these philosophical issues, but content ourselves with a more prosaic approach. How should a planner, whose job is to respect public ownership of some productive assets, in conjunction with private ownership of some inputs, try to allocate resources? We limit ourselves to a static, complete information world. The dynamic question of investment, and issues of asymmetric information and incentive compatibility -- issues that have been the focus of the planning literature -- are not our concern here. (How the planner might implement the various proposals that we discuss is the subject of another paper (Roemer [in press].)

Our interest **in** this question is in large part motivated by the economic reforms that are **being introduced in** socialist countries. Many observers take the introduction of markets in socialist countries as equivalent to the introduction of capitalism. We think this hasty conclusion is based upon the view that public ownership is synonymous with central allocation. We view central allocation as one way of possibly implementing a regime of public ownership; there may be more decentralized ways of implementing public ownership, including the use of markets. But to study this

2. DESIDERATA OF PUBLIC OWNERSHIP

The problem is most simply posed by considering an economic environment defined as $\xi = \langle \omega_{11}, \omega_{21}, \ldots, \omega_{M1}; U_1, U_2, \ldots, U_M; f \rangle$, consisting of a population of M persons, the i^{th} one of whom is endowed with an amount ω_{i1} of some homogeneous private good x_1; a production function $f(x_1)$ which represents a technology capable of transforming the private good x_1 into an output of another good x_2; and ordinal preferences of the people for the goods (x_1, x_2) represented by utility functions U_i. We postulate that the good x_1 is privately owned by the individuals, and that they publicly own the technology f. (The function f may already incorporate publicly owned productive resources.) This specification is special in its assumption that there is only one privately owned good, which has both a use in consumption and production, and one other consumption good, produced on the publicly owned technology. (We relax these restrictions in Section 7.)

One interpretation is that the privately owned good is labor[1]. If agents have different skill levels, they can be represented by different endowments ω_{i1} of the input. Thus skill differences in this model take the form of different capacities to supply labor, which can be commonly measured in efficiency units.

We propose four desiderata of a solution that implements public ownership of the technology f, in conjunction with private ownership of the endowments ω_{i1} of the good x_1, in economic environments of the type ξ.

(1) Efficiency, in the sense of Pareto optimality;

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[1]It is on occasion natural to view a person's labor time as non-transferable to other persons. This can be modelled by postulating that the consumption set is a subset of the non-negative orthant defined by certain inequalities. Of course, a particular solution concept may single out different allocations in economies that differ only in the consumption sets of their persons. because the set of feasible (or efficient) allocations may well differ. Our analysis covers all these possibilities, because it applies to arbitrary consumption sets. The solution concepts that we propose are, in particular, always well-defined, at least under convexity assumptions (see Section 7 below).

(2) equal division of benefits from use of the publicly owned technology;

(3) distribution of the output X_2. produced from the public technology, in proportion to individual contributions of the input (labor) in production;

(4) universal gain from improvements in the publicly owned asset.

Desiderata (2),(3) and (4) will be embodied in precise formulations below. We will insist on desideratum (1) throughout. For the domain of economic environments that we shall consider, no more than one of (2), (3), and (4) is generally compatible with (1). This leads to the consideration of three pairs of desiderata, (1) and (2), (1) and (3). and (1) and (4). Indeed, each such pair will characterize one proposal for public ownership.

We maintain that these desiderata are suggested by the public and private property rights we are assuming. Equal benefit means equal gain from the initial endowment-, gains from the use of the public technology should be the same for all. Distribution according to individual contribution is a way of recognizing each person's right to a return to her privately owned input. That all should (weakly) gain from an improvement in the publicly owned technology seems a natural requirement of public ownership. This requirement also states that the welfare of no person should rise if the publicly owned technology deteriorates in quality.

All of (1) through (4) characterize the behavior of the most natural solution to the public-private ownership problem on the class of linear economic environments where the production function takes the form $f(x_1)=\beta x_1$, for some $\beta\geq 0$. One can consider our analysis as an attempt to preserve these four properties, which the linear economies enjoy, on larger domains of environments. For a linear economy, denoted by $\langle\omega_{11},...\omega_{M1}; U_1,...,U_M; \beta\rangle$, consider the 'autarkic' allocation which results from each person having free access to the technology, and using it as much as he pleases. The autarkic allocation $((x_{11},x_{12}),...,(x_{M1},x_{M2}))$ is defined as the solution of the programs, for each i:

$$\max U_i(x_{i1},x_{i2})$$

$$\text{s.t. } x_{i2}= \beta(\omega_{i1}-x_{i1}) . \qquad (2.1)$$

This solution naturally implements a conception of public ownership of the technology in conjunction with private ownership of the input, because in the linear economies, there are no positive or negative externalities from joint use of the technology. Public ownership suggests, in this case, free access to the technology; private ownership of x_1 requires allowing each to use her private endowment as she chooses.

This autarkic solution for the linear economies satisfies desiderata (1) through (4). Clearly it is Pareto optimal. It has the equal benefit property. Benefits for person i are the difference between the value of the final consumption vector (x_{i1},x_{i2}), and the value of the initial endowment vector $(\omega_{i1},0)$. Because we have Pareto optimality , a natural valuation for these vectors is the vector of efficiency prices. If we normalize using commodity x_1 as numéraire, the vector of efficiency prices is (under differentiability):

$$(1,p) = (1, (\partial U_i/\partial x_{i2})/(\partial U_i/\partial x_{i1})) , \qquad (2.2)$$

which , when f'>0, can also be written as (1,p) = (1,f'). Hence, equality of benefits requires that:

$$\text{for all } i, h \qquad px_{i2}+(x_{i1}-\omega_{i1}) = px_{h2}+ (x_{h1}-\omega_{h1}) . \qquad (2.3)$$

In our case, $(1,p)=(1,1/\beta)$, and (2.3) is satisfied because it follows from the budget constraint (2.1) that:

$$\text{for all } i, h \qquad x_{i2}+\beta(x_{i1}-\omega_{i1}) = x_{h2}+ \beta(x_{h1}-\omega_{h1}) .$$

See §3.1 below for alternative interpretations of the equal benefit condition.

With respect to desideratum (3), notice that output is distributed in proportion to input contributed to production because for all i:

$$(\omega_{i1}-x_{i1})/ x_{i2} = 1/\beta ,$$

i.e., a person contributes exactly the amount of labor embodied in her consumption of x_2.

We make desideratum (4) precise by requiring that, as the technology improves in the sense that the technological parameter β increases to β', the allocation assigned in the economy $\langle\omega_{11},\ldots\omega_{M1};U_1,\ldots,U_M;\beta'\rangle$ should (weakly) increase the utility of every agent from what it was when the technology was β. This is clearly so for the autarkic solution, as an increase of β to β' is equivalent to a fall in the relative price of output.

Therefore , the four desiderata are simultaneously satisfied in the case of a linear technology. Another special case in which they are simultaneously satisfied is the case of identical persons (same endowments of x_1 and preferences). There is, in this case, a symmetric allocation that satisfies all requirements (see proof of theorem 4 below).

In the next three sections, we discuss solution concepts on a domain of economies of this prototype that preserve, along with Pareto optimality, desiderata (2), (3), and (4), respectively.

3. THE EQUAL BENEFIT SOLUTION

3.1 BASIC IDEA

Consider the prototypical economic environment $\xi=\langle\omega_{11},\ldots,\omega_{M1};U_1,\ldots,U_M;f\rangle$, where the production function f may now display variable returns to scale. Productive efficiency now requires:

$$\Sigma_h x_{h2} = f(\Sigma_h\omega_{h1} - \Sigma_h x_{h1}) . \qquad (3.1)$$

As introduced in Section 2, the equal benefit approach views public ownership as requiring that all persons benefit equally from the presence of a publicly owned resource or technology, i.e., condition (2.3) must be satisfied in addition to (3.1). One can interpret the solution in three different scenarios.

(A) <u>A market for good x_2</u>

From (2.3) and (3.1), one immediately obtains:

$$x_{i1} + p\, x_{i2} = \omega_{i1} + \pi/M , \qquad (3.2)$$

where $\pi = p\,(f(\Sigma_h\omega_{h1} - \Sigma_h x_{h1})) - (\Sigma_h\omega_{h1} - \Sigma_h x_{h1})$ are the profits of the firm that produces and sells x_2.

Thus, when f is concave, the solution can be viewed as resulting from the operation of markets when the income of the consumers includes an equal share in the profits of a competitive firm.

If, on the other hand, returns to scale are increasing, the firm is publicly owned and distributes equally among consumers the losses derived from marginal cost pricing.

(B) <u>Cooperative production</u>

Interpret x_1 as leisure, and let $L_i = \omega_{i1} - x_{i1}$ be the amount of labor that person i contributes to a cooperative production process that yields x_2. Write $L = \Sigma_i L_i$. Now $\Sigma_i x_{i2} = f(L)$ and (3.2) becomes:

$$px_{i2} = L_i + (p\ f(L) - L)/M,$$

i.e., $$x_{i2}/f(L) = L_i/pf(L) + 1/M - L/pf(L)M,$$

i.e., $$x_{i2}/f(L) = LL_i/Lpf(L) + (1/M)(1 - L/pf(L)),$$

or: $$x_{i2}/f(L) = \lambda\ L_i/L + (1-\lambda)\ (1/M), \quad \text{where } \lambda = L/pf(L). \qquad (3.3)$$

i.e.,person i's share in the cooperative output is a linear combination of her labor share and the "equal division" share 1/M, where the coefficient λ of her labor share is the ratio of average cost to marginal cost. Hence, as noted in Section 2 above, i's share in output coincides with her share in labor if there are constant returns to scale or if her labor contribution is the average one (i.e., if $L_i = L/M$) but not otherwise.

(C) <u>External diseconomies or economies</u>

We consider diseconomies first. Think of the "commons", where the citizens take their cows to pasture, or, better, a lake where individual fishermen spend time (good x_1) and obtain fish (good x_2). There is no trade in fish. Production is carried out individually and each fisherman eats his catch. All fishermen are equally good and equally lucky: if the amounts of time spent are $(L_1,\ldots,L_M)$, then i gets $(f(L)/L)L_i$ units of fish. We assume that the average catch, f(L)/L, is decreasing in L. The <u>laissez faire</u> solution is given by:

$$\text{for } i = 1,\ldots, M,\ x_{i1} \text{ maximizes } U_i(x_{i1},\ (f(L)/L)(\omega_{i1} - x_{i1})).$$

where $L = \Sigma_h (\omega_{h1} - x_{h1})$.

As is well known, the laissez faire solution is inefficient. This fact, sometimes called hyperbolically the "tragedy of the commons", is on occasion marshalled to advocate the privatization of public property: see Weitzman [1974] and Roemer [in press] for criticisms of this view. An efficiency inducing mechanism consistent with public ownership is the Pigovian tax-subsidy scheme. Pigovian taxes can be understood as either paid in commodity 1 and proportional to the individual amount of x_2 obtained, or as paid in commodity 2 and proportional to the individual amount of labor spent.

The Pigovian scheme consists of imposing individual tax rates t_i on the amount of time spent and lumpsum subsidies S_i, all paid in fish (smoked?), such that the government budget balances and an efficient allocation is attained as each fisherman chooses x_{i1} in order to:

$$\text{Max } U_i(x_{i1}, S_i + ((f(L)/L) - t_i)(\omega_{i1} - x_{i1})).$$

The first order condition is:

$$\partial U_i/\partial x_{i1} - (\partial U_i/\partial x_{i2})((f(L)/L) - t_i) = 0,$$

i.e, $$t_i = (f(L)/L) - (\partial U_i/\partial x_{i1})/(\partial U_i/\partial x_{i2}) ,$$

which, together with the efficiency condition:

$$(\partial U_i/\partial x_{i1})/(\partial U_i/\partial x_{i2}) = f' ,$$

yields : $$t_i = (f(L)/L) - f',$$

i.e., $t_i = t_h = t$ (same tax rate for everybody). The total tax revenue, equal to the total subsidy, is $tL = \Sigma_i S_i = f(L) - f'L$.

The final consumption vector of fisherman i is $(x_{i1}, S_i + f' \cdot (\omega_{i1} - x_{i1}))$. For the final state to satisfy the equal benefit property it must be true that (writing the price vector as $(1, 1/f')$) for all fishermen i and h:

$$x_{i1} - \omega_{i1} + (S_i/f') + \omega_{i1} - x_{i1} = x_{h1} - \omega_{h1} + (S_h/f') + \omega_{h1} - x_{h1},$$

i.e., $$S_i = S_h = (f(L) - f' L)/M.$$

i.e., at the equal benefit solution the revenue created by the Pigovian tax is equally distributed among fishermen, independently of the number of hours that each spends fishing, and, hence, of their individual tax burden.

When the externality is positive, $f(L)/L$ is increasing in L. The Pigovian scheme is then a subsidy rate s and an individualized lumpsum tax T_i. Person i's final consumption vector is $(x_{i1}, -T_i+((f(L)/L)+s)(\omega_{i1}-x_{i1}))$. Now, at the equal benefit solution, each person pays the same lumpsum tax T independently of the amount of time worked.

To conclude, let the technology adopt the limit form $f(x_1)=\omega_2$, i.e., x_2 is a non-produced commodity initially available in ω_2 publicly owned units. We call this the 'manna economy.' Efficiency now requires:

$$\sum_h x_{h1}=\sum_h \omega_{h1},$$
$$\sum_h x_{h2} = \omega_2, \qquad (3.4)$$

which, together with (2.2-3) yields $x_{i1}+px_{i2}=\omega_{i1}+p\omega_2/M$. Because $p=(\partial U_i/\partial x_{i2})/(\partial U_i/\partial x_{i1})$, the vector (x_{i1},x_{i2}) maximizes i's utility subject to the budget constraint that i faces when she initially owns ω_{i1} of the first good and ω_2/M of the second one. One can thus visualize this solution as resulting from the following process: the public authority first distributes the available resource ω_2 equally among the citizens who then trade it among themselves in competitive markets.

3.2 INADEQUACY OF THE EQUAL BENEFIT SOLUTION

These interpretations lead us to view the equal benefit solution as inadequate for capturing the notion of public ownership. It rather seems to reflect "equal private ownership" or "syndicalism." We review the meaning of the solution in the cases presented above.

(i) Production under decreasing returns: a market for x_2

The equal benefit solution is indistinguishable from the Walrasian equilibrium in which each initially owns a 1/M share of the firm (or of the scarce resource that is at the root of the decreasing returns).

(ii) Production under increasing returns: a market for x_2

Now the losses from marginal cost pricing are distributed irrespective of consumption. A person who does not care about commodity 2 must suffer a decrease in her private wealth just because other people like x_2 and she is forced to "own" a share of the technology. (Such a person would indeed be better off were she to abandon society.)

(iii) Cooperative production

The amount of output obtained is not in line with the amount of effort put into production. Output is instead distributed according to a particular weighted sum (with endogenous weights) of the amount of effort provided and the equal share 1/M, see (3.3) above. Schemes of the type

$$x_{i2} / f(L) = \alpha L_i/L + (1-\alpha) (1/M),$$

where α is usually some exogenous parameter, are found in the literature on cooperatives, see, e.g., Sen (1966), Israelsen (1980) and Kang (1987). These schemes are sometimes justified as a combination of the principles "to each according to his work" (this is reflected in the term $\alpha L_i/L$) and "to each according to his needs" (the term $(1-\alpha)(1/M)$). But this interpretation of the second term presupposes that everybody's needs are alike, and if every person is identical then there is a symmetric allocation that satisfies all three solution concepts studied in this paper.

Israelsen (1980) defines the polar case $\alpha = 0$ as the "commune", and the other extreme, $\alpha = 1$, as the "collective." The equal benefit solution is an intermediate case, with an endogenous α (see (3.3)). The alternative notion of proportional solution (see Section 4 below) corresponds to the pure "collective."

(iv) Externalities

The Pigovian scheme is an instrument to implement efficiency. But here it is also a redistributive mechanism.

(v) Manna economy

Manna, in principle collectively owned, is first privatized and then traded against x_1, a transferable good. The solution is indistinguishable from the competitive allocation when each initially owns ω_2/M units of manna: the benefits of public ownership are actually the trade gains from these equal endowments. Consider the limit case where consumer 1 does not care about manna. She will still benefit from manna provided that somebody else likes manna.

4. THE PROPORTIONAL SOLUTION

The discussion above has pointed towards the need to distinguish "public ownership" from "equal private ownership." Private ownership includes the right to use, destroy and prevent other people from using, as well as the right to transfer these rights. By "use" we mean "enjoy, benefit from", perhaps without destroying or depleting. One can benefit from, say, a pure public good without depleting it . Hence, in some cases, the right to use can be transferred without transferring the right to destroy or deplete. Also, the right to use can in principle be transferred without transferring the right to transfer the right to use (e.g.. usufruct). This suggests that, when something is publicly owned: (a) nobody has the right to destroy it; (b) everybody has the right to **use** it, if the good can be used without depletion.

Public property rights can in principle be transferred to individuals, and. indeed, some rights must be if what is publicly owned is not a pure public good. But there are two levels of transference, (i) Privatization occurs when the individual acquires full ownership-- not only the right to consume but also the right to sell; (ii) allocation of use occurs, on the other hand, when the individual acquires only the right to use, but not full ownership (e.g., she does not acquire the right to sell). Conversely, private property rights can in principle be transferred to society, e.g., by sale or by expropriation.

These ideas reflect G.A. Cohen's [1986] distinction between something not being owned by anybody ("res nullius"), in which case anybody can privatize it, and being publicly owned, in which case privatization requires the explicit consent of society.

The proportional solution tries to capture this basic distinction between private and public forms of ownership. We wish to illustrate it with some particular examples before offering a general definition.

(i) Production of a private good

First we consider our prototypical economy. The proportional solution prohibits transfers of the privately owned good x_1 that are unrelated to the consumption of the publicly produced commodity. We require that i's contribution to the cost of production of x_2 equal the amount of x_1 embodied in i's consumption of x_2. If total production of x_2 is $\Sigma_h x_{h2}$ and i's consumption of x_2 is x_{i2} then the proportional solution is given by Pareto optimality and the condition:

$$\omega_{i1} - x_{i1} = \frac{\Sigma_h(\omega_{h1} - x_{h1})}{\Sigma_h x_{h2}} x_{i2} . \qquad (4.1)$$

The reader is referred to Mas-Colell and Silvestre (in press, Section IV.2) for an alternative derivation of this condition. We now apply this solution concept to the interpretations of the production economy presented in Section 3.2 above.

(A) <u>A market for good x_2: the Berkeley Coop</u>

Good x_2 is produced and sold at marginal cost, p, by a publicly owned firm. This generates, under decreasing returns to scale, profits in the amount $\pi = p\Sigma_h x_{h2} - \Sigma_h(\omega_{h1} - x_{h1})$. Condition (4.1) can be written as:

$$x_{i1} + p\, x_{i2} = \omega_{i1} + \left(p - \frac{\Sigma_h(\omega_{h1} - x_{h1})}{\Sigma_h x_{h2}}\right) x_{i2} \quad ,$$

i.e.,

$$x_{i1} + p\, x_{i2} = \omega_{i1} + \theta_i \pi,$$

where

$$\theta_i = x_{i2} / \Sigma_h x_{h2} \quad .$$

i.e., profits are distributed among consumers in proportion to their purchases of commodity x_2. This scheme reminds the authors that, when buying at the Berkeley Coop, the cash register records the buyer's membership number, and any operating profits that may appear at the end of the year are then distributed among members in proportion to their purchases.

Under increasing returns to scale marginal cost pricing induces losses that are then charged to consumers in proportion to their purchases. In either case net payment equals average cost times quantity.

(B) Cooperative production

Write, as in Section 3 above, $L_i = \omega_{i1} - x_{i1}$, the amount of labor that i contributes, and $L = \Sigma_h L_h$. Equation (4.1) becomes:

$$L_i = \frac{L}{f(L)} x_{i2} \quad ,$$

i.e., $x_{i2}/f(L) = L_i/L$.

The proportional solution is a Pareto optimal allocation in which output is distributed according to labor contribution. This implements the principle "to each according to his work." It is the pure "collective" scheme in Israelsen's [1980] terms.

(C) Externalities

Now production is carried out individually and there is no market. Consider first the negative externality case. As in Section 3.1, the Pigovian efficiency tax is:

$$t = (f(L)/L) - f', \tag{4.2}$$

and the individualized subsidies must satisfy the government's budget constraint: $\Sigma_h S_h = tL$. Person i's net final consumption of x_2 is (see Section 3.1. above) $x_{i2} = S_i + f' L_i$. Now (4.1) becomes:

$$L_i = \frac{L}{f(L)} (S_i + f' L_i) \quad ,$$

i.e.,

$$S_i = \left(\frac{L}{f(L)} - f' \right) L_i \quad .$$

or, using (4.2),

$$S_i = t\,L_i \quad ,$$

i.e., person i gets back, as a lumpsum subsidy, exactly the amount that he paid as Pigovian tax. Contrary to what happened with the equal benefit solution, the Pigovian scheme has no redistributive implication: it is here just an instrument for efficiency.

If the externality is positive, the Pigovian scheme consists of a subsidy rate s and lumpsum tax T_i (see Section 3.1 above). The proportional solution again effects no redistribution, i.e., $T_i = sL_i$.

(ii) The manna economy

The proportional solution efficiently allocates the publicly owned manna without altering the distribution of the privately owned good, i.e., (i) it does not impose transfers of the good x_1 while allocating the rights to use x_2 (no expropriation), and (ii) it does not allow for private exchange of the two goods (no privatization). The proportional solution is here defined by the condition that the M-tuple of consumption vectors (ω_{i1}, x_{i2}) be a Pareto optimal allocation. See Figure 1 for M = 2. In the particular case where only person 1 likes good x_2 she gets all the manna.

(iii) A non-produced pure public good initially available in ω_2 units

Our last example involves a pure public good. This is at variance with the rest of the present paper, which focuses on private goods[2]. But it illustrates the allocation of use without privatization discussed above at the start of Section 4: use

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2Our analysis may be extended to the public good case by following Arrow's (1969) technique of interpreting a public good as M private goods. (If the public good is a produced one then the corresponding M private goods are perfectly complementary outputs.) The concept of the proportional solution actually coincides, under some conditions, with the notion of Balanced Linear Cost Share Equilibrium presented in Mas-Colell and Silvestre (in press) as a formalization of Lindahl's (1919) solution for economies with public goods. Mas-Colell and Silvestre (in press, Section IV.31) covers also purely private goods as well as the intermediate case of externalities in consumption. Their approach could be used to extend the notion of the proportional solution to the latter case.

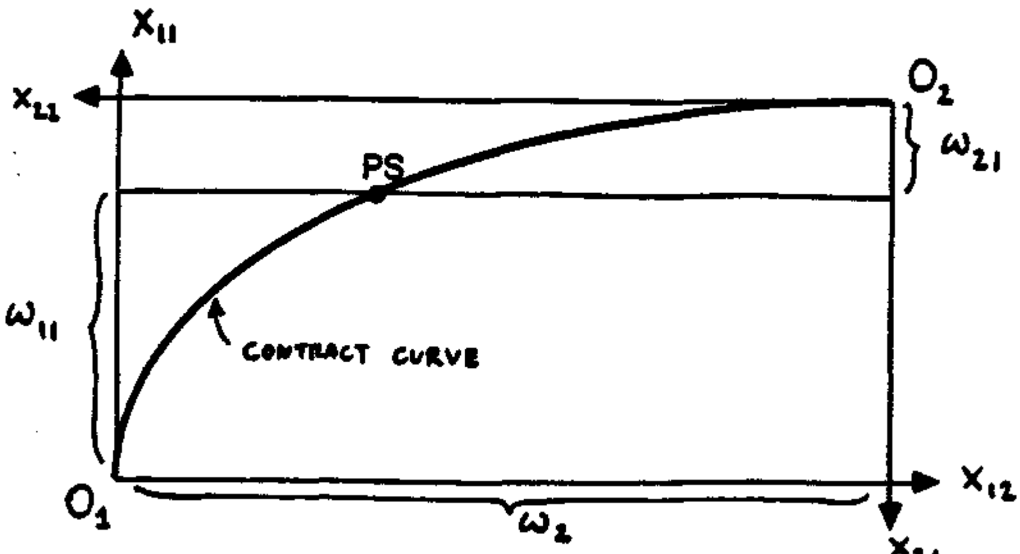

Figure 1 The proportional solution in the manna economy is point PS

can be isolated from the transfer of property rights because use does not require depletion.

Let x_2 be a nonproduced public good initially available in ω_2 units (i.e., $f(x_1) = \omega_2$ for all x_1) and let the endowment ω_2 be publicly owned. The proportional solution is here a direct translation of the idea that everybody has the right to use the good and, hence, nobody has the right to prevent other people from using the good. This solution here requires:

$$(x_{i1}, x_{i2}) = (\omega_{i1}, \omega_2) \quad , \; i = 1,\dots,M.$$

i.e., it permits no transfers of the privately owned good x_1.

Note the sharp contrast with the notion of equal benefit, which here requires that, for any pair of agents i,h,

$$x_{i1} - \omega_{i1} + p_i \cdot \omega_2 = x_{h1} - \omega_{h1} + p_h \cdot \omega_2,$$

where $p_\ell = (\partial U_\ell / \partial x_2)/(\partial U_\ell / \partial x_{i1})$, $\ell = i,h$,

i.e., equal benefit requires agents with higher marginal valuation for the public good to transfer some privately owned good x_1 to those with low valuation.

A general definition of the proportional solution (for private goods) is offered in Section 7.3 below. There we prove existence under convexity assumptions, i.e., we show that one can generally find an allocation that is efficient and satisfies a condition like (4.1). Our existence theorem, moreover, allows for a variety of assumptions on the supply of labor and on the transferability of labor time among individuals because it covers arbitrary closed and convex consumption sets.

5. TECHNOLOGICAL MONOTONICITY AND THE CONSTANT-RETURNS-EQUIVALENT MECHANISM

Public ownership suggests that no person should suffer when the publicly owned technology becomes more abundant or productive. While the notion of equal benefits requires each person to gain equally from the initial to final allocation in a given economy, the notion of universal gain described in desideratum (4) requires everyone to benefit by virtue of a change of a certain type in the economic environment, namely, an increase in the endowment of the publicly owned asset. We model this with the axiom of technological monotonicity, which was introduced in Roemer (1986), and has been studied in several other papers (Moulin and Roemer, in press, Moulin, 1987).

Before defining technological monotonicity, we adopt a somewhat different vantage point from that of §§3,4 . We return to the prototypical economic environment of §2. Fix the parameters $(\omega_{11},\omega_{21},\ldots,\omega_{M1},U_1,\ldots U_M)=(\omega,U)$ and consider three domains of economic environments, defined by: $\Sigma^{DR}(\omega,U)=$ $\{\xi=\langle\omega_{11},\ldots,\omega_{M1};U_1,\ldots,U_M;f\rangle \mid f\varepsilon\ DR\}$; $\Sigma^{IR}(\omega,U) = \{\xi=\langle\omega_{11},\ldots,\omega_{M1};U_1,\ldots,U_M;f\rangle \mid f\varepsilon\ IR\}$; and $\Sigma(\omega,U) = \Sigma^{DR}(\omega,U) \cup \Sigma^{IR}(\omega,U)$, where DR $=\{f \mid f(x_1)/x_1$ is non-increasing$\}$ and IR$=\{f \mid f(x_1)/x_1$ is non-decreasing$\}$. In addition, define the sub-domain of constant-returns economies as $\Sigma^{CR}(\omega,U) = \{\xi\varepsilon\Sigma(\omega,U) \mid f(x_1)=\beta x_1$, for some $\beta>0\}$. If we now vary the parameters (ω,U), we define, for fixed M:

$$\Sigma = \cup\ \Sigma(\omega,U) ,$$

where the union is defined for all non-negative vectors $\omega\varepsilon\ R^M$ and for all profiles U in which the M utility functions represent arbitrary, continuous, strictly monotonic preferences. Σ^{DR} and Σ^{IR} are likewise defined as the domains of decreasing returns and increasing returns environments, respectively.

<u>Definition</u>. An <u>allocation mechanism</u> is a function $F : \Omega \to R^{2M}$, where Ω is some domain of economic environments. F associates to each $\xi\varepsilon\ \Omega$ an allocation[3] in ξ.

We state technological monotonicity as a property of an allocation mechanism defined on one of the domains Σ, Σ^{DR} or Σ^{IR}. Let the allocation mechanism F assign bundle $F_i(\xi)= (x_{i1},x_{i2})$ to person i. Define the mapping induced by the mechanism into utility space :

$$U(F(\xi)) = (U_1(F_1(\xi)),\ \ldots,\ U_M(F_M(\xi))).$$

---

[3]More generally, an allocation mechanism can be a correspondence, so long as the induced utility mapping, defined below, is single-valued.

<u>Technological Monotonicity.</u> (TMON) Let $\xi^1=\langle\omega_{11}, \ldots, \omega_{M1};U_1,\ldots,U_M;f\rangle$ and $\xi^2=\langle\omega_{11},\ldots\omega_{M1};U_1,\ldots,U_M;g\rangle$ be two economic environments (in $\Sigma(\omega,U)$) such that for all x_1, $f(x_1)\geq g(x_1)$. The allocation mechanism F is technologically monotonic if $U(F(\xi^1))\geq U(F(\xi^2))$.

Neither the equal benefit solution of §3 nor the proportional solution of §4 satisfies TMON[4]. The equal benefit solution is seen to violate TMON by the following example. Consider an economy with two people, who have initial endowments ω_{11} and ω_{21} of the private good x_1. The equal benefit solution assigns to each person a .5 share in the profits of the public firm. Suppose that the first person has preferences $U(x_1,x_2) = x_1$ and that the technology is linear. There are zero profits at equilibrium. Because the first person's budget constraint is $x_{11} + px_{12} = \omega_{11}$ and he does not care for the x_2 good, his final consumption, is $(\omega_{11},0)$. Now suppose the technology deteriorates to some concave production function everywhere dominated by the previous linear technology. In the new economy there will be positive profits; the first person's consumption will increase to $(\omega_{11}+.5\Pi, 0)$ where Π is total profits. Thus the equal benefit solution violates technological monotonicity.

It is only slightly more difficult to describe an example showing that the proportional solution (PS) also violates TMON. Consider an economic environment with two persons and with the linear technology $f_1(x_1)=\beta x_1$. The second person's preferences are $U_2(x_{21},x_{22})=x_{22}$; the first's preferences are $U_1(x_{11},x_{12})=(\beta-\varepsilon)x_{11}+x_{12}$ -- he is willing to trade 1 unit of good one for $\beta-\varepsilon$ units of good 2, where ε is small and positive. With the technology f_1, both persons contribute all of their input to production and consume only good 2. Now consider a technological change to a

[4]Indeed, TMON is only well-defined for allocation mechanisms that are single-valued in utility space, which the equal benefit solution and the proportional solution are not. Even when these solutions are single-valued for a pair of economic environments that differ only in their technologies, however, TMON need not hold.

production function f_2 which is piece-wise linear; for $x_1<.9\omega_{21}$, f_2 has a slope much steeper than β, and for $x_1>.9\omega_{21}$, f_2 has a slope $\alpha<\beta-\varepsilon$, and furthermore, $f_2\geq f_1$ for the relevant domain of x_1. With technology f_2, person 2 still puts all his input into production; this guarantees that the marginal rate of transformation will be α, which is too small to induce person 1 to contribute any of his endowment of good 1 to production. Thus person 1, who contributes no input, gets no share of the profits under PS. His utility, in the second environment, at the PS is $U_1(\omega_{11},0)$. Because the allocation $(\omega_{11},0)$ was available to him in the first environment, but he did not choose it, his utility there was greater than in the second environment under PS. Hence PS violates TMON for the first person.

What allocation mechanisms satisfy TMON, are Pareto optimal, and assign on the sub-domain Σ^{CR} the autarkic allocation? (The autarkic allocation for linear economies was defined in §2; we argued that it was the clear implementation of public ownership on linear economies.) We state this last condition as an axiom:

<u>Free Access in Linear Economies</u> (FALE) Let $\xi\varepsilon\ \Sigma^{CR}$. Then $F(\xi)$ is the autarkic solution.

<u>Theorem 1</u>. <u>There is a unique allocation mechanism defined on the domain</u> Σ <u>(alternatively,</u> Σ^{DR} <u>or</u> Σ^{IR} <u>) that satisfies Pareto optimality, TMON and FALE</u>. <u>We call it the 'constant-returns-equivalent (CRE)' mechanism</u>.

<u>Proof</u>: Appendix.

The CRE mechanism is described as follows. Consider any economic environment $\xi^*=\langle\omega;U;f\rangle$ and the associated domain of constant returns economies $\Sigma^{CR}(\omega,U)$. By FALE, F is defined on $\Sigma^{CR}(\omega,U)$, and $U(F(\xi))$ traces out a strictly monotone increasing path in utility space as ξ ranges over the domain $\Sigma^{CR}(\omega,U)$. There exists a unique linear economic environment ξ in $\Sigma^{CR}(\omega,U)$ such that $U(F(\xi))$ is on the Pareto frontier of ξ^*. $F(\xi^*)$ is defined as the (or any one of the) allocations in ξ^* which induces that utility allocation. The name "CRE" describes the property that the mechanism chooses, for any environment ξ^*, an allocation which is Pareto optimal and Pareto indifferent to what the same persons, defined by (ω, U), could achieve by free access in <u>some</u> linear economy. See Figure 2 for a representation of CRE in a manna economy.

The CRE solution arises in other contexts. It was studied by Mas-Colell (1980) and Moulin(1987), and Moulin and Roemer(in press). Indeed, Theorem 1 is only a slightly reformulated version of Moulin's(1987) Theorem 1, and his subsequent discussion in that paper.

6. SIMULATIONS

We have highlighted three solution concepts : the equal benefit solution (EBS), the proportional solution (PS), and the constant-returns-equivalent mechanism (CRE). Each coincides with the autarkic solution for the prototypical linear economies. Each chooses an efficient, symmetric allocation (perhaps among others) when all individuals are identical. This section reports on simulations which give some feeling for the behavior of these solutions.

Our 'empirical work' consisted of computing each solution for various parameter values of the two person economic environment $\langle \omega_{11}, \omega_{21}; U_1, U_2; f \rangle$. The parameters of the economy are as follows:

$$U_1(x_{11}, x_{12}) = x_{11}^{b} x_{12}^{1-b} ,$$

$$U_2(x_{21}, x_{22}) = x_{21}^{c} x_{22}^{1-c},$$

$$\omega_{11} \ \varepsilon \ [0,1],$$

$$\omega_{21} = 1-\omega_{11},$$

$$f(x) = (.5x)^{a}.$$

An economy of this type is specified by the vector $\gamma = (a, b, c, \omega_{11})$. We simulated the three solution concepts for twenty-two choices of γ.

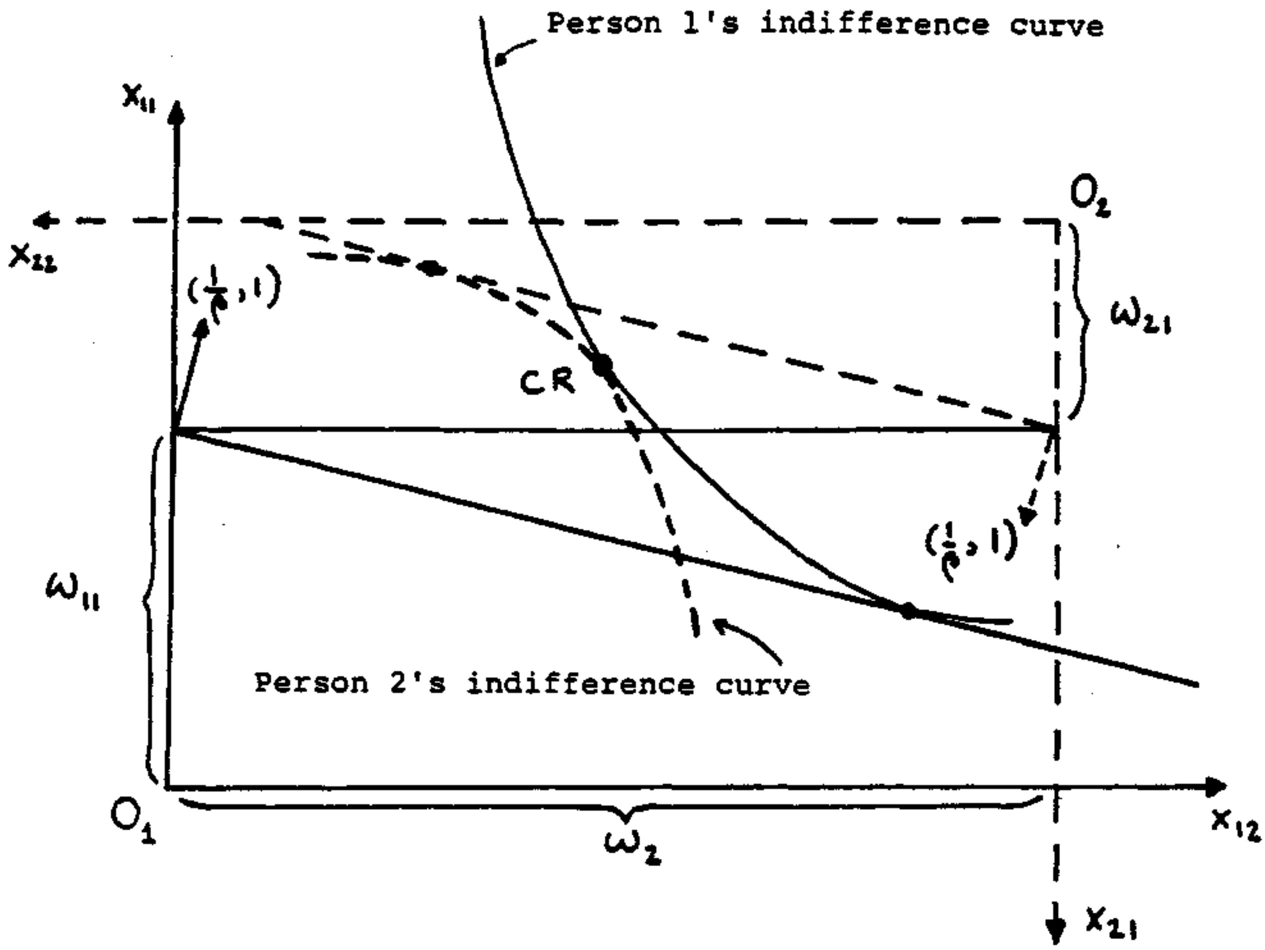

FIGURE 2

THE CRE MECHANISM CHOOSES POINT CR
IN THE MANNA ECONOMY
(Each person reaches the utility she would reach under autarky and the constant return technology $x_{i2} = \beta(\omega_{i1} - x_{i1})$.)

The striking observation was the similarity between CRE and PS when the allocations produced by the two mechanisms are graphed in utility space[5] . (We did obtain an example in which the PS violates TMON.) CRE and PS differ significantly from EBS. This is most easily seen when b is taken to be close to zero and c close to 1. The first person can be called Industrious, as his relative preference for output over leisure will induce him to work long hours, while the second person can be called Lazy, due to her preference for leisure. Suppose Lazy and Industrious have equal initial endowments of labor. Under CRE and PS, Lazy either gains very little in final utility, or loses a bit, as the technology improves. But under EBS, she gains significantly from the technological improvement. This is because she shares equally in the profits, which are produced mostly by the labor of Industrious on the public technology. One feels that under EBS Lazy is exploiting Industrious. Indeed, if we view a person as exploited when his share in total labor supplied is greater than his share of output, as suggested by the Marxian definition, then Industrious is consistently exploited by Lazy in EBS. The solution which, by definition, avoids exploitation is the PS: for in the PS each agent consumes an amount of output that embodies exactly the amount of input that he contributes to production, where the amount of input embodied in a share o* of the output is defined to be equal to o* times the input expended in producing the total output.

7. ECONOMIES **WITH MANY INPUTS AND PRIVATE** FIRMS

7.1 A **GENERAL ECONOMY**

The purpose of this section is to present definitions of the equal benefit and propositional solutions in general economic environments, and to prove the existence of these solutions under standard convexity assumptions.

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[5]We checked the robustness of the similarity of the PS and CRE solutions by simulating economies where u_1 and U_2 (a) CES utility functions, with independently varying parameters, (b) quasi-linear utility functions, both linear in the output, (c) quasi-linear utility functions, with one linear in the output and the other linear in the input. In each of these cases, the similarity of the two solutions holds.

There are N commodities partitioned into two groups: the private sector commodities, indexed 1 to K, and the public sector commodities, indexed K+1 to N. The defining distinction is that commodities in $\{1,\ldots,K\}$ are either initially owned by individuals or produced by privately owned firms, but neither collectively owned nor produced by a publicly owned firm.[6]

There are F privately owned firms, indexed 1 to F, and one publicly owned firm ("the public productive sector"), with production sets $Y_1,\ldots,Y_F$ and Y_C, respectively ("C" for "collectively owned"). We assume:

A.1 If $y_f \in Y_f$ and $k \in \{K+1,\ldots,N\}$, then $y_{fk} \leq 0$, $f=1,\ldots F$.

A.2 If $y_C \in Y_C$ and $k \in \{1,\ldots,K\}$, then $y_{Ck} \leq 0$.

A.3 For $g=1,\ldots,F$, C, Y_g is closed, displays free disposal (i.e., $Y_g - R_+^N \subset Y_g$) and satisfies $Y_g \cap R_+^N = \{0\}$.

A.4 If $y_g \in Y_g$, $g=1,\ldots F$, C, and $\sum_{f=1}^{F} y_f + y_C \geq 0$, then $y_g = 0$, $g=1,\ldots,F$, C.

There are M persons. Person i is endowed with an initial vector $\omega_i \in R_+^N$ and nonnegative shares d_{if} in the profits of private firm f, $f=1,\ldots,F$, satisfying: $\sum_i d_{if} = 1$. The next assumption agrees again with our partition of the commodity space.

A.5 If $k \in \{K+1,\ldots,N\}$, then $\omega_{ik} = 0$, $i=1,\ldots,M$.

We assume that person i's consumption set X_i is a closed and convex subset of R_+^N. The reader is referred to Arrow-Hahn (1971, Chapter 4) and Newman (1987) for interpretations. We note that the description of the consumption set may capture not only constraints on the combinations of labor services that an individual may supply

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[6] This partition of the commodity space entails no loss of generality, since one can still cover economies with, say, a good that is produced by both public and private firms by introducing two perfect substitutes, one in $\{1,\ldots,K\}$ and the other one in $\{K+1,\ldots,N\}$.

but also on the amounts of leisure that a person may enjoy. As Arrow-Hahn (1971, A.4.2, p. 77), we assume:

A.6 For $i=1,\ldots,M$, there exists a consumption vector $\overline{x}_i \in X_i$ such that $\overline{x}_i \leq \omega_i$ and $\overline{x}_{ij} < \omega_{ij}$ if $\omega_{ij} > 0$.

Person i's preferences are represented by a continuous, concave, nondecreasing and non-satiated utility function $U_i : X_i \to R$ that agrees with the convention:

A.7 For $i=1,\ldots,M$, $U_i(\overline{x}_i) = 0$.

The public sector initially owns the nonnegative vector ω_C, satisfying:

A.8 If $k \in \{1,\ldots,K\}$ then $\omega_{Ck} = 0$.

We also assume that:

A.9 There exists a $\overline{y} \in \sum_{f=1}^{F} Y_f + Y_C$ such that $\overline{y} + \sum_{i=1}^{M} \omega_i + \omega_C$ is strictly positive.

<u>Definition</u>. An <u>allocation</u> is vector $s = (x_1,\ldots,x_M, y_1,\ldots,y_F,y_C) \in R^{N(M+F+1)}$ such that: (i) $x_i \in X_i$, $i=1,\ldots,M$;

(ii) $y_g \in Y_g$, $g=1,\ldots,F$, C.

Denote by S the set of allocations. Write

$$\tilde{z}: S \to R^N : \tilde{z}(s) = \sum_{i=1}^{M} x_i - \sum_{f=1}^{F} y_f - y_C - \sum_{i=1}^{M} \omega_i - \omega_C .$$

<u>Definition</u>. An allocation s is <u>feasible</u> if $\tilde{z}(s) \leq 0$.

Denote by $\hat{S}$ the set of feasible allocations.

<u>Assumption A.10</u>. $\hat{S}$ is compact.

<u>Definition</u>. An allocation $s \in \hat{S}$ is <u>Pareto optimal</u> if there does not exist another feasible allocation s' such that: $U_i(x_i') \geq U_i(x_i)$, $i=1,\ldots,M$, with strict inequality for at least one i.

The notions of equal benefit solution and proportional solution were defined in the prototypical economy by the conditions of Pareto optimality and an extra condition involving the (2 dimensional) vector of efficiency prices. Efficiency prices will play an even more important role in the generalized definition of this Section since they will be used for aggregating inputs or outputs.

Because no differentiability assumptions are imposed in this Section and because production sets may in principle be nonconvex, the definition of efficiency prices must appeal to elaborate mathematical concepts. Guesnerie (1975) originally used the polar of the cone of interior displacements; here we will insteady apply the more recent notion of Clarke's normal cone to a production set Y_g at point y_g, that we denote $N(Y_g, y_g)$, see Clarke (1983) and Cornet (1987) for definitions and interpretations. When Y_g has a smooth boundary, $N(Y_g, y_g)$ is simply the gradient of the production function defining such boundary. When Y_g is convex and possible nondifferentiable, $N(Y_g, y_g)$ is simply the set of price vectors for which y_g maximizes profits over Y_g, i.e.,

$$N(Y_g, y_g) = \{p \in R_+^N \mid p.y_g \geq p.y_g' \text{ for all } y_g' \in Y_g\}$$

see Clarke (1983, Proposition 2.44, p. 52) and Cornet (1987, Remark 2.2, p. 4). One can visualize $N(Y_g, y_g)$ in general as the set of convex combinations of vectors that are orthogonal to Y_g at y_g.

Definition. The nonzero vector $p^* \in R_+^N$ is a vector of efficiency prices for the Pareto optimal allocation s^* if

(a) $p^* \cdot x_i^* \leq p^* \cdot x_i$ for all $x_i \in X_i$ such that $U_i(x_i) \geq U_i(x_i^*)$, $i=1,\ldots,M$;

(b) $p^* \in N(Y_g, y_g^*)$ for $g = 1,\ldots, F, C$, where $N(Yg, y_g^*)$ is Clarke's normal cone to Y_g at y_g^*, and

(c) $p^* \cdot \bar{z}(s^*) = 0$.

The existence theorems for the equal benefit solution and for the proportional solution will require assumptions like the ones used in proving the existence of competitive equilibrium. We shall in particular need to guarantee that expenditure minimization implies utility maximization. For this purpose we shall use Arrow-Hahn's (1971, p. 117-118) "indirect resource relatedness." It is not the simplest alternative, but it is the most adequate for our problem. On the one hand, because the public sector commodities K+1,...,N are not initially owned by any particular person one cannot use an assumption like "there is an x_i^o in X_i such that $x_{ij} < \omega_{ij}$ for all j." (See Debreu, 1959, Section 5.7). On the other hand, an assumption like "$X_i = R_+^N$ and U_i is strictly increasing in all its arguments" could not cover situations where leisure time is assumed to be non-transferrable among consumers, see footnote 1 above.

<u>Definition</u>. Person i' is <u>resource related</u> to person i" if for every feasible allocation s there exists an allocation s' and a vector $\Delta\omega$ such that:

(a) $\sum_{i=1}^{M} x_i' \leq \sum_{f=1}^{F} y_f' + y_C' + \sum_{i=1}^{M} \omega_i + \omega_C + \Delta\omega$,

(b) $U_h(x_h') \geq U_h(x_h)$, h=1,...,M ,

(c) $U_{i''}(x_{i''}') > U_{i''}(x_{i''})$,

(d) $\Delta\omega \geq 0$,

(e) $\Delta\omega_j > 0$ only if $\omega_{i'j} > 0$.

In words, i' is resource related to i" if by increasing the endowments of some of the commodities that i' initially owns it is possible to make i" better off without making anybody worse off.

<u>Definition</u>. Person i' is <u>indirectly resource related</u> to person i" if there is a (n+1) sequence of persons h_ν, ν=0,...,n, with h_0 = i', h_n = i" and h_ν resource related to $h_{\nu+1}$, (ν=0,...,n-1).

<u>Assumption A.11</u>. Every person is indirectly resource related to every other person.

Special existence theorems will cover the extreme case where all consumers are identical.

Definition. All consumers are identical if they have the same preferences, represented by the utility function $\hat{U}$, and the same initial endowments, denoted by ω_I, and if $d_{if} = 1/M$, for $i=1,\dots,M$, $f=1,\dots,F$.

7.2 A GENERAL FORMULATION OF THE EQUAL BENEFIT SOLUTION

Definition. An allocation s^* is an equal benefit solution if:

(i) s^* is Pareto optimal;

(ii) There exists a vector of efficiency prices p^* for s^* such that, for any pair of consumers i, h,

$$p^*\cdot x_i^* - p^*\cdot\omega_i - \sum_{f=1}^{F} d_{if}\, p^*\cdot y_f^* = p^*\cdot x_h^* - p^*\cdot\omega_h - \sum_{f=1}^{F} d_{hf}\, p^*\cdot y_f^* .$$

Remark. Note that the prototypical and manna economies of §§3-6 are special cases of the one in §7.1, and that this general definition agrees with out previous descriptions of the equal benefit solution.

Our next Theorem and the proof of Theorem 3 make precise the equivalence of the equal benefit solution and the Walrasian solution for equal endowments of public resources and equal shares in the profit of the public sector.

Theorem 2. Assume that for $g=1,\dots,F$, C, Y_g is convex. Let s^* be an equal benefit solution with associated efficiency prices p^*. Then:

(i) for $g=1,\dots F$, C, y_g^* maximizes $p^*\cdot y_g$ on Y_g.

(ii) for $i=1,\dots,M$, x_i^* maximizes $U_i(x_i)$ subject to:

$$p^*\cdot x_i = p^*\cdot\omega_i + \sum_{f=1}^{F} d_{if}\, p^*\cdot y_f^* + \frac{1}{M}[p^*\cdot y_C^* + p^*\cdot\omega_C].$$

Proof. Appendix.

Theorem 3. Let Y_g be convex, $g=1,\dots,F$, C. Then an equal benefit solution exists.

Proof. Easy consequence of the existence of competitive equilibrium for the private ownership economy $\hat{\mathcal{E}}$ where $\hat{\omega}_i = \omega_i + (1/M)\,\omega_C$ and where consumers own equal shares of the firm labelled C, i.e., $\hat{d}_{iC} = 1/M$.

Theorem 3 covers convex technologies only. The presence of increasing returns to scale is sometimes the rationale for public ownership of firms. The question of existence for nonconvex technologies is, however, beyond the scope of this paper. But we wish to emphasize that the concept is in principle applicable to nonconvex technologies. We offer, to this end, an existence theorem for the special case of identical consumers without restrictions on the technology.

Theorem 4. If all consumers are identical then a symmetric equal benefit solution exists.

Proof: Appendix.

7.3 A GENERAL FORMULATION OF THE PROPORTIONAL SOLUTION

To motivate our next definition, rewrite (4.1) above as:

$$x_{i1} = \omega_{i1} + \frac{x_{i2}}{\Sigma_h x_{h2}} \Sigma_h (x_{h1} - \omega_{h1}). \tag{7.1}$$

i.e., i's consumption of the privately owned good equals his endowment minus a fraction $x_{i2}/\Sigma_h x_{h2}$ of the inputs (costs) of the public sector. The fraction $x_{i2}/\Sigma_h x_{h2}$ is the ratio of i's consumption of public sector output to the total output of the public sector.

Here the vectors of net outputs and inputs of the public sector are defined as follows. Given a feasible allocation s, define, for $j \in \{1,\ldots,N\}$.

$$t_j^+ = \max\,\{0,\ \sum_{i=1}^{M} x_{ij} - \sum_{i=1}^{M} \omega_{ij} - \sum_{f=1}^{F} y_{fj}\}\,,$$

$$t_j^- = \min\,\{0,\ \sum_{i=1}^{M} x_{ij} - \sum_{i=1}^{M} \omega_{ij} - \sum_{f=1}^{F} y_{fj}\}\,.$$

and write: $t^+(s) = (t_1^+(s),\ldots,t_N^+(s))$ and $t^-(s) = (t_1^-(s),\ldots,t_N^-(s))$. One can interpret $t^+(s)$ (resp. $t^-(s)$) as the vector of net quantities that the public sector delivers to (resp. obtains from) the private sector. These quantities will be aggregated by means of a vector of efficiency prices.

Let consumer i own private firm f. If firm f uses inputs from the public sector, then the value of such inputs should be included in i's use of public sector goods. This motivates the following general version of the ratio $x_{i2}/(\sum_h x_{h2})$ in (7.1).

__Definition__. __Given__ $(p,s) \in R_+^N \times \hat{S}$, i's __proportional share__ $\theta_i(p,s)$ is

$$\theta_i(p,s) = \begin{cases} \dfrac{\sum_{j=K+1}^{N} p_j x_{ij} - \sum_{f=1}^{F} d_{if} \sum_{j=K+1}^{N} p_j y_{fj}}{p\cdot t^+(s)}, & \text{if } p\cdot t^+(s) > 0, \\ 1/M, & \text{if } p\cdot t^+(s) = 0. \end{cases}$$

Clearly $\theta_i(p,s) \geq 0$ since for $j \in \{K+1,\ldots,n\}$, $-y_{fj} \geq 0$. Moreover, feasibility implies that $t_j^+(s) + t_j^-(s) \leq y_{cj} + \omega_{cj}$. Hence, if $t_j^+(s) > 0$, then $y_{cj} + \omega_{cj} > 0$, which, by our assumption on the partition of commodities, implies that $j \in \{K+1,\ldots,N\}$. Therefore, $\sum_{i=1}^{M} \theta_i(p,s) = p\cdot t^+(s)/p\cdot t^+(s) = 1$, i.e., the definition of $\theta_i(p,s)$ guarantees __bona fide__ shares.

Second, we add to person i's initial endowment ω_i of private sector goods the (net) output of such goods produced by the firms that i owns. This leads to:

__Definition__. An allocation s^* is a __proportional solution__ if:

(i) s^* is Pareto optimal;

(ii) there exists a vector of efficiency prices p^* for s^* such that for every consumer i,

$$\sum_{j=1}^{K} p_j^* x_{ij}^* = p^* \cdot \omega_i + \sum_{f=1}^{F} d_{if} \sum_{j=1}^{K} p_j^* y_{fj}^* + \theta_i(p^*, s^*)\, p^* \cdot t^-(s^*).$$

Theorem 5. Let Y_g be convex, $g=1,\ldots,F, C$. Then a proportional solution exists.

Proof. Appendix.

Our next Theorem makes precise the equivalence of the proportional solution and the Walrasian solution when the endowments of public resources and shares in the profits of the public firm are proportional to the consumption of public sector commodities.

Theorem 6. (A). If $(p^*, s^*) \in R_+^N \times \hat{S}$ satisfies:

(i) for $f=1,\ldots,F, C$, y_g^* maximizes $p^* \cdot y_g$ on Y_g,

(ii) for $i=1,\ldots,M$, x_i^* maximizes $U_i(x_i)$ subject to:

$$p^* \cdot x_i = p^* \cdot \omega_i + \sum_{f=1}^{F} d_{if} p^* \cdot y_f^* + \theta_i(p^*, s^*)\, [p^* \cdot y_C^* + p^* \cdot \omega_C],$$

then s^* is a proportional solution with efficiency prices p^*.

(B). The converse of (A) is true if Y_g is convex, $g=1,\ldots,F, G$.

Proof. Appendix.

As in § 7.2, we study the identical consumer case with the aim of providing a special existence result for nonconvex technologies.

Theorem 7. Let consumers be identical. Then a symmetric proportional solution exists.

Proof. Appendix.

It may be of independent interest to establish that, in the identical consumer case, an equal benefit solution is basically a proportional solution. (The converse is, however, false.)

Theorem 8. Let all consumers be identical, assume that $0 \in X_i$, $i \in \{1,\ldots,M\}$ and let s^* be an equal benefit solution. Then there exists a proportional solution $\hat{s}$ such that $\hat{U}(\hat{x}_i) = \hat{U}(x_i^*)$, $i = 1,\ldots, M$.

Proof. Appendix.

7.4 DIFFICULTIES IN GENERALIZING THE CONSTANT-RETURN-EQUIVALENT MECHANISM

How does the axiomatic characterization of the constant-returns-equivalent mechanism fare in economic environments with many commodities? With several private inputs and one publicly produced output, the FALE axiom can be reformulated to specify the autarkic solution when the technology is linear, not only constant-returns. (With several inputs, it is only on the linear technologies that no producer creates any externalities for others.) TMON is reformulated to require that if the production set in one economy includes the production set from another, then all agents (weakly) benefit in the more 'advanced' economy.

Theorem 9. For the domain of economic environments with some fixed number of private inputs greater than one, there is no allocation mechanism which satisfies Pareto optimality, TMON and FALE.

Proof. Appendix.

Thus the CRE mechanism does not generalize to the case of many inputs as do the previous solutions. Furthermore, in the several input case the motivation of TMON as a requirement of public ownership in the presence of private ownership of factor inputs dissolves. For suppose your endowment of two private inputs is $(\omega_{11}, \omega_{12})$ and mine is $(\omega_{21}, \omega_{22})$, where ω_{22} is much larger than ω_{21}. Suppose, in the first technology, the input x_2 is very important, but after the technological inprovement, x_2 becomes almost useless in production. If private ownership of inputs is respected, there is no reason to require that my welfare improve under the allocation mechanism which respects the required property rights. I may lose by virtue of the devaluation in my endowment of the privately owned good. This problem did not arise with a single input which varied only in one dimension (say, skill level, but not type of skill).

In the case, however, where there are several private inputs but all individuals are identical, TMON continues to be a salient requirement for public ownership: the above critique does not apply. Furthermore, on the domain of economic environments consisting of identical individuals, the CRE mechanism is characterized by Pareto optimality, TMON and FALE, and it chooses a symmetric allocation.

An impossibility result also obtains for the case of many outputs and a single input.

8, CONCLUSION

Of the three solutions, we believe that the proportional solution best implements public ownership of some resources in the presence of privately owned inputs. The equal benefit solution is an interesting point of reference, but we think it is more in the spirit of equal private ownership than of collective ownership. It is especially hard to justify when production is characterized by increasing returns, as it inflicts losses upon people who do not consume the publicly produced good.

Technological monotonicity is an appealing notion, but it is probably too strong a requirement to insist upon for public ownership in the general case, as the CRE mechanism does not generalize to the several commodity case. Even in the case of purely private ownership, a similar monotonicity condition is too strong. The utilities arising from Walrasian equilibrium are not monotonic in the endowments of the agents, but that fact does not cause us to view markets as transgressions of private ownership.

We must, in any event, emphasize the appropriate jurisdiction for these models. "Primary' goods, such as education or health, are not financed according to use in countries where their availability is taken to be a right. (In Canada and Great

Britain, for instance, the national health service is supported out of general revenues.) In addition, distributional concerns may motivate a government to support a public transportation system, for example, by a property tax which falls mainly on people who never use the subway. We do not impugn these public policies. Our concern has been to distinguish conceptually between public and private ownership of economic resources.

APPENDIX: PROOFS

Definition: Let F be an allocation mechanism defined on a domain of economic environments Ω. F is a monotone utility path (MUP) mechanism on Ω if there is a fixed montone path in utility space such that for any $\xi\varepsilon\Omega$, the induced utility allocation $u(F(\xi))$ lies on the path.

Lemma 1. *Let F be an allocation mechanism defined on Σ (respectively, Σ^{DR} or Σ^{IR}) which satisfies Pareto optimality (PO) and TMON. Then F is a monotone utility path mechanism on each subdomain $\Sigma(\omega,U)$. The same is true if Σ is replaced by Σ^{DR} or Σ^{IR}.*

Proof. It suffices to show that for any ξ^1, ξ^2 ε $\Sigma(\omega,U)$, either $U(F(\xi^1)) \geq U(F(\xi^2))$ or $U(F(\xi^2)) \geq U(F(\xi^1))$. If $\xi^1 = (\omega;U;f)$ and $\xi^2 = (\omega;U;g)$, define:

$$h(x_1) = \max\,(f(x_1),\, g(x_1)),$$

and, noting that h is an admissible production function, consider $\xi^* = (\omega;U;h)$ ε $\Sigma(\omega,U)$. If ξ^1, ξ^2 ε Σ^{DR} (or Σ^{IR}) then h inherits the DR (or IR) property from f and g, and so ξ^* ε Σ^{DR} (Σ^{IR}, respectively). Since $h \geq f$ and $h \geq g$, it follows by TMON that $U(F(\xi^*)) \geq U(F(\xi^1))$ and $U(F(\xi^*)) \geq U(F(\xi^2))$. But by definition of h, $F(\xi^*)$ is feasible for at least one of ξ^1 or ξ^2 -- say ξ^1. Then by PO, it follows that $U(F(\xi^*)) = U(F(\xi^1))$, and so $U(F(\xi^1)) \geq U(F(\xi^2))$.

Lemma 2. *If F satisfies FALE on $\Sigma(\omega,U)$, then F is a monotone utility path mechanism on $\Sigma^{CR}(\omega,U)$.*

Proof: By FALE, F chooses the autarkic allocation on linear economies, whose production functions are $f_\beta(x_1) = \beta x_1$. Define $\xi_\beta = (\omega;U;f_\beta)$. Since U is strictly monotonic, $U(F(\xi_\beta))$ is strictly monotone increasing in all its components with respect to increases in β.

Proof of Theorem 1: (after Moulin, 1987). It is easy to verify that the Constant-Returns-Equivalent mechanism satisfies the three axioms. We prove that only the CRE mechanism satisfies them. For any (ω,U), by Lemma 1, F is a monotone utility path mechanism on $\Sigma(\omega,U)$. But the path is determined by the behavior of F on the linear economies, by Lemma 2. $F(\xi)$ is therefore the allocation determined by the intersection of ξ's utility frontier with the monotone path determined by the linear economies. Note that all points on the monotone path Pareto dominate $(U_1(\omega_{11},0), U_2(\omega_{21},0))$, and also that any such allocation must be (strictly) Pareto optimal in ξ: so $F(\xi)$ is PO. It follows that F on $\sum(\omega,U)$ is the Constant-Returns-Equivalent mechanism.

__Remark__. The same proof suffices for F defined on $\sum^{DR}(\omega,U)$ and $\sum^{IR}(\omega,U)$, since the classes of DR and IR production functions are closed under the binary max operation. Note this is also true for the class of convex production functions (and so Theorem 1 is true on that domain), but it is not true for the class of concave production functions.

We turn now to the economy of Section 7.

__Definition__. A __utility allocation__ is an M-vector $u = (u_1,\ldots,u_M)$ such that $U_i(x_i) = u_i$ for some $x_i \in X_i$, $i=1,\ldots,M$.

__Definition__. Given a utility allocation u, the set of __u-feasible allocations__ is:

$$\hat{W}(u) = \{s \in \hat{S} \mid U_i(x_i) \geq u_i,\ i=1,\ldots,M\}.$$

A utility allocation u is __feasible__ if there exists a feasible allocation s such that $U_i(x_i) = u_i$, $i=1,\ldots,M$. A feasible utility allocation u is __weakly Pareto optimal__ if there does not exist another feasible utility allocation u' such that $u_i' > u_i$, $i=1,\ldots,M$.[7] Denote by $\overline{U}$ the set of weakly Pareto optimal utility allocations.

__Definition__. The nonnegative price vector $p \in R^N_+$, $p \neq 0$, __supports the utility allocation__ u if $p\cdot\bar{z}(s) \geq 0$ for all allocations satisfying $U_i(x_i) \geq u_i$, $i=1,\ldots,M$.

__Lemma 3__. __Assume that__ Y_g __is convex__, $g=1,\ldots,F, C$. __Let__ u __be a weakly Pareto optimal utility allocation__. __Then there exists a__ $p \in R^N_+$ __that supports__ u.

__Proof__: Standard; see, e.g., Arrow-Hahn (1971, Th. 4.4(a)-(b), p. 93).

[7] Note that, contrary to standard usage, Arrow-Hahn (1971) write "Pareto efficient" for "weakly Pareto optimal."

Lemma 4. Assume that Y_g is convex, $g=1,\ldots,F, C$. Let $p^* \in R^N_+$, support the weakly Pareto optimal utility allocation u^*, and let $s^* \in \hat{W}(u^*)$. Then:

(i) p^* is a vector of efficiency prices for s^*.

(ii) $p^* \cdot y^*_g \geq p^* \cdot y_g$ for all $y_g \in Y_g$, $g=1,\ldots,F, C$.

Proof: See Arrow-Hahn (1971, Th. 4.4(d), p. 93). As noted in 7.1 above, when Y_g is convex, the condition $p^* \in N(Y_g, y^*_g)$ is equivalent to profit maximization.

Lemma 5. If $(p^*, s^*) \in R^N_+ \times \hat{S}$ satisfies:

(i) $p^* \neq 0$,

(ii) $p^* \cdot y^*_g \geq p^* \cdot y_g$ for all $y_g \in Y_g$, $g=1,\ldots,F, C$,

(iii) $p^* \cdot \tilde{z}(s^*) = 0$,

then $p^* \cdot x^*_h > 0$ for some h.

Proof: By (ii), (iii) and Assumption A.9 and (i) we have that

$$p^* \cdot \sum_{i=1}^{M} x^*_i = p^* \cdot \left[\sum_{i=1}^{M} \omega_i + \omega_C + \sum_{f=1}^{F} y^*_f + y^*_C \right] \geq p^* \cdot \left[\sum_{i=1}^{M} \omega_i + \omega_C + \overline{y} \right] > 0.$$

Lemma 6. Let $p^* \in R^N_+$ support the weakly Pareto optimal utility allocation u^*, let $s^* \in \hat{W}(u^*)$ and assume that for some consumer i, $p^* \cdot x^*_i > 0$ and $p^* \cdot (x^*_i - \omega_i) \geq 0$.

Then x^*_i maximizes U_i subject to $p^* \cdot x_i \leq p^* \cdot x^*_i$.

Proof. We use the proof of Arrow-Hahn (1971, Lemma 5.1, p. 108) to show that $p^* \cdot x^*_i > p^* \cdot \overline{x_i}$ and hence, since $\overline{x_i} \in X_i$, that x^*_i does not minimize $p^* \cdot x_i$ on X_i. Utility maximization will then follow from standard arguments (Debreu, 1959, (1) of 4.9, p. 69 or Arrow-Hahn, 1971, Lemma 4.3, p. 81).

By Assumption A.6, $p^*\cdot(\omega_i-\overline{x_i}) \geq 0$ and $p^*\cdot(\omega_i-\overline{x_i}) > 0$ if and only if $p^*\cdot\omega_i > 0$. Hence, since "$p^*\cdot x_i^* > 0$" implies that either "$p^*\cdot\omega_i > 0$" or $p^*\cdot(x_i^*-\omega_i) > 0$," we have that "$p^*\cdot x_i^* > 0$ implies that either "$p^*\cdot(\omega_i-\overline{x_i}) > 0$ and (by hypothesis) $p^*\cdot(x_i^*-\omega_i) \geq 0$" or "$p^*\cdot(x_i^*-\omega_i) > 0$ and $p^*\cdot(\omega_i-\overline{x_i}) \geq 0$." In either case, $p^*\cdot(x_i^*-\overline{x_i})$ is the sum of a positive term and a nonnegative term, and hence positive. Thus, $p^*\cdot x_i^* > p^*\cdot\overline{x_i}$.

__Lemma 7__. __Let__ Y_g __be convex__, $g=1,\dots,F, C$. __Let__ $p^* \in R^N_+$ __support the weakly Pareto optimal utility allocation__ u^* __and let__ $s^* \in \hat{W}(u^*)$. __If__ i' __is indirectly resource related to__ i^*, $p^*\cdot x_{i^*}^* > 0$, __and__ $p^*\cdot x_{i'}^* \geq p^*\cdot\omega_{i'}$, __then__ $p^*\cdot x_{i'}^* > 0$.

__Proof__. (We adapt the proof of Arrow-Hahn, 1971, Lemma 5.4 and Corollary 5.1, p. 118-119). We first consider the case where i' is (directly) resource related to i^*. By A.11, there exists an allocation s' and a vector $\Delta\omega$ such that:

(a) $\sum x_h' \leq \sum y_f' + y_C' + \sum\omega_h + \omega_C + \Delta\omega$,

(b) $U_h(x_h') \geq U_h(x_h^*)$, $h \in \{1,\dots,M\}$,

(c) $U_{i^*}(x_{i^*}') > U_{i^*}(x_{i^*}^*)$,

(d) $\Delta\omega \geq 0$,

(e) $\Delta\omega_j > 0$ only if $\omega_{i'j} > 0$.

Because p^* supports u^*, (b) implies, by Lemma 4(i), that $p^*\cdot x_h' \geq p^*\cdot x_h^*$, $h \in \{1,\dots,M\}$. Moreover, by Lemma 6, $x_{i^*}^*$ maximizes U_{i^*} subject to $p^*\cdot x_{i^*} \leq p^*\cdot x_{i^*}^*$. Therefore, by (c), $p^*\cdot x_{i^*}' > p^*\cdot x_{i^*}^*$. It follows that $p^*\cdot\sum x_h' > p^*\cdot\sum x_h^*$.

Because Y_g is convex, y_g^* maximizes $p^*\cdot y_g$ on Y_g, $g=1,\dots,F, C$, (Lemma 4 (ii)), i.e.,

$$p^* \cdot \left(\sum_{f=1}^{F} y'_f + y'_C \right) \leq p^* \cdot \left(\sum_{f=1}^{F} y^*_f + y^*_C \right).$$

Because $p^* \cdot \tilde{z}(s^*) = 0$, we have that

$$p^* \cdot \sum_{h=1}^{M} x^*_h = p^* \cdot \left[\sum_{f=1}^{F} y^*_f + y^*_C \right] + p^* \cdot \left[\sum_{h=1}^{M} \omega_h + \omega_C \right].$$

Thus,

$$p^* \cdot \sum_{h=1}^{M} x'_h > p^* \cdot \left[\sum_{f=1}^{F} y'_f + y'_C \right] + p^* \cdot \left[\sum_{h=1}^{M} \omega_h + \omega_C \right].$$

But from (a) and because $p^* \in R^N_+$, we have that

$$p^* \cdot \sum_{h=1}^{M} x'_h \leq p^* \cdot \left[\sum_{f=1}^{F} y'_f + y'_C \right] + p^* \cdot \left[\sum_{h=1}^{M} \omega_h + \omega_C + \Delta\omega \right]$$

i.e., $p^* \cdot \Delta\omega > 0$, and thus, $p^*_j \Delta\omega_j > 0$ for some j. But by (e) this implies that $p^*_j \omega_{i'j} > 0$, i.e., $p^* \cdot x^*_{i'} \geq p^* \cdot \omega_{i'} \geq p^*_j \omega_{i'j} > 0$. This concludes the proof for the case where i' is (directly) resource related to i''. Repetition of the argument yields the result for the assumption of indirect resource relatedness.

<u>Proof of Theorem 2</u>. As noted in section 7.1 above, (i) follows from the convexity of Y_g. Moreover, from (ii) in the definition of an equal benefit solution we have that, for i=1,...,M,

$$p^* \cdot x^*_i - p^* \cdot \omega_i - \sum_{f=1}^{F} d_{if}\, p^* \cdot y^*_f = \frac{1}{M} p^* \cdot \left(\sum_{h=1}^{M} x^*_i - \sum_{h=1}^{M} \omega_h - \sum_{f=1}^{F} y^*_f \right) = \frac{1}{M} p^* \cdot (y^*_C + \omega_C),$$

the last equality following from the definition of efficiency prices, i.e.,

$$p^* \cdot x^*_i = p^* \cdot \omega_i + \sum_{f=1}^{F} d_{if}\, p^* \cdot y^*_f + \frac{1}{M} \left[p^* \cdot y^*_C + p^* \cdot \omega_C \right].$$

which in particular implies that $p^* \cdot x_i^* \geq p^* \cdot \omega_i$, $i \in \{1,\ldots,M\}$, because maximum profits are nonnegative. By Lemma 5, $p^* \cdot x_h^* > 0$ for some h, and thus, by A.11 and Lemma 7, $p^* \cdot x_i^* > 0$ for $i \in \{1,\ldots,M\}$. Lemma 6 in turn guarantees utility maximization.

<u>Proof of Theorem 4</u>. Define $\hat{Y} = \frac{1}{M}(\sum_{f=1}^{F} Y_f + Y_C)$. Consider the maximization problem: Choose $(x^*, \hat{y})$ in order to maximize $\hat{U}(x)$ subject to:

$$x = \omega_I + \frac{1}{M}\omega_C + y,\ y \in \hat{Y}.$$

By Assumption A.10, the constraint set of this problem is compact, and thus a solution $(x^*, \hat{y})$ exists. By the definition of $\hat{Y}$, $\hat{y} = \frac{1}{M}(\sum_{f=1}^{F} y_f^* + y_C^*)$ for some $(y_1^*,\ldots,y_F^*, y_C^*)$ such that $y_g^* \in Y_g$, $g=1,\ldots,F, C$. We want to show that $s^* = (\overbrace{x^*,\ldots,x^*}^{M \text{ vectors}},\ y_1^*,\ldots,y_F^*, y_C^*)$ is an equal benefit solution. Efficiency is easily proved by contradiction. Let s' Pareto dominate s^*. Then it is easy to show, by the concavity of $\hat{U}$, that

$$\hat{U}\left(\frac{1}{M}\sum_{i=1}^{M} x_i^{\cdot}\right) > \hat{U}(x^*),$$

and that the vector $(\frac{1}{M}\sum_{i=1}^{M} x_i^{\cdot},\ \frac{1}{M}[\sum_{f=1}^{F} y_f^{\cdot} + y_C^{\cdot}])$ satisfies the constraints of the above maximization problem, contradicting the assumption that $(x^*,\hat{y})$ solves it. Thus, s^* is efficient. By Cornet (1987, Theorem 3.3., p. 6) a vector of efficiency prices p^* exists. Moreover, by the definition of (x^*, y^*), we have that:

$$x^* = \omega_i + \frac{1}{M}\omega_C + \frac{1}{M}(\Sigma\, y_f^* + y_C^*)\ .$$

i.e., $$p^{*}\cdot x^{*} - p\cdot\omega_{i} - \frac{1}{M}p^{*}\cdot\sum_{f=1}^{F} y_{f}^{*} = \frac{1}{M}p^{*}\cdot\left[\omega_{C} + y_{C}^{*}\right],$$

proving that condition (i) in the definition of an equal benefit solution is also satisfied.

Lemma 8. Assume that Y_g is convex, $g=1,\ldots,F$, C. Then the correspondence $P: \overline{U} \to \Delta^{N-1}$: $P(u) = \{p \in \Delta^{N-1} \mid p \text{ supports } u\}$, (where $\overline{U}$ is the set of weakly Pareto optimal utility allocations and $\Delta^{N-1} = \{p \in R_{+}^{N} \mid \sum_{j=1}^{N} p_j = 1\}$) is upper-hemicontinuous with non-empty, compact and convex values.

Proof. Arrow Hahn (1971, Theorem 4.6, p. 99; the nonemptiness of $P(u)$ for $u \in \overline{U}$ is guaranteed by Lemma 3.)

Lemma 9. Let Y_g be convex, $g=1,\ldots,F$, C. Write $U' = \overline{U} \cap R_{+}^{M}$ for the set of weakly Pareto optimal, nonnegative utility allocations, and Δ^{M-1} for the simplex $\{v \in R_{+}^{M} \mid \sum_{i=1}^{M} v_i = 1\}$.

(A) The mapping: $\hat{v}$: $U' \to \Delta^{M-1}$: $\hat{v}(u) = (1/\sum_{h} u_h)\, u$, is a homeomorphism; denote its inverse by $\hat{u}$.

(B) The correspondence: $\hat{W}$: $U' \to \hat{S}$ is upper-hemicontinuous, with nonempty, convex and compact values.

(C) $\hat{S}$ is convex.

Proof: **(A)**: Arrow-Hahn (1971, Section 5.2, p. 111-114). **(B)**: Arrow-Hahn (1971, Section 5.3, p. 114). **(C)**: Arrow-Hahn (1971, Theorem 4.2, p. 89).

Lemma 10. If $s \in \hat{S}$, then for any $p \in R_{+}^{N}$ and for $i=1,\ldots,M$,

$$\sum_{j=K+1}^{N} p_j x_{ij} - \sum_{f=1}^{F} d_{if} \sum_{j=K+1}^{N} p_j y_{fj} - \theta_i(p,s)\; p\cdot t^{+}(s) = 0.$$

<u>Proof</u>. If $p \cdot t^{+}(s) > 0$, then the equation follows immediately from the definition of $\theta_i(p,s)$. So let $p \cdot t^{+}(s) = 0$, which in particular implies that $\sum_{j=K+1}^{N} p_j \cdot t_j^{+}(s) = 0$. Because $\omega_{ij} = 0$ for $j \in \{K+1,\dots,N\}$, this can be rewritten:

$$\sum_{j=K+1}^{N} p_j \left(\sum_{i=1}^{M} x_{ij} - \sum_{f=1}^{F} y_{fj} \right) = 0.$$

But each term in the sum is nonnegative, because $-y_{fj} \geq 0$ for $f \in \{1,\dots,F\}$ and $j \in \{K+1,\dots,N\}$. Hence, $p_j x_{ij} = p_j y_{fj} = 0$ for $f \in \{1,\dots,F\}$ and $j \in \{K+1,\dots,N\}$.

<u>Proof of Theorem 5</u>. We adapt to our problem Arrow-Hahn's (1971, Chapter 5) proof of existence of competitive equilibrium. First, define Θ: $\Delta^{N-1} \times \hat{S} \to R^M$:

$$\Theta(p,s) = \begin{cases} \theta(p,s) = (\theta_1(p,s),\dots,\theta_M(p,s)), \text{ if } p \cdot t^{+}(s) > 0. \\ \Delta^{M-1} \text{ otherwise.} \end{cases}$$

Because θ is continuous and single valued on $\{(p,s) \in \Delta^{N-1} \times \hat{S} \mid p \cdot t^{+}(s) > 0\}$, Θ is upper hemicontinuous and convex valued on its domain $\Delta^{N-1} \times \hat{S}$, which is mapped by Θ into Δ^{M-1}.

Second, define V: $\Delta^{N-1} \times \Delta^{M-1} \times \hat{S} \to \Delta^{M-1}$;

$V(p, \theta, s) = \Delta^{M-1} \cap \{v \in R^M \mid v_i = 0 \text{ if } \sigma_i(p, \theta, s) < 0\}$,

where $\sigma_i(p, \theta, s) = p \cdot \omega_i + \sum_{f=1}^{F} d_{if} \sum_{j=1}^{K} p_j y_{fj} - \sum_{j=1}^{K} p_j x_{ij} + \theta_i p \cdot t^{-}(s)$.

Clearly, $V(p, \theta, s)$ is a compact and convex set. It is nonempty, because of the definition of $t^{-}(s)$ and the fact that $\sum \theta_i = 1$,

$$\sum_{i=1}^{M} \sigma_i(p, \theta, s) = 0, \tag{9.1}$$

and therefore $\sigma_h(p, \theta, s) \geq 0$ for some h.

We claim now that V is upper-hemicontinuous.

Let: $\{p^\nu, \theta^\nu, s^\nu\} \to (p, \theta, s)$, $(p^\nu, \theta^\nu, s^\nu) \in \Delta^{N-1} \times \Delta^{M-1} \times \hat{S}$,

$$v^\nu \to v \text{ , } v^\nu \in V(p^\nu, \theta^\nu, s^\nu),$$

and let $\sigma_i(p, \theta, s) < 0$. Because σ_i is continuous, $\sigma_i(p^\nu, \theta^\nu, s^\nu) < 0$ for ν large enough, i.e., $v_i^\nu = 0$ for ν large enough and hence $v_i = 0$, i.e., $v_i = 0$ whenever $\sigma_i(p, \theta, s) < 0$ and therefore $v \in V(p, \theta, s)$. We conclude that V is upper-hemicontinuous.

Consider now the correspondence

$$\Phi: \Delta^{N-1} \times \Delta^{M-1} \times \Delta^{M-1} \times \hat{S} \to \Delta^{N-1} \times \Delta^{M-1} \times \Delta^{M-1} \times \hat{S},$$

$$\Phi(p, v, \theta, s) = P(\hat{u}(v)) \times V(p, \theta, s) \times \Theta(p, s) \times \hat{W}(\hat{u}(v)),$$

an upper-hemicontinuous and nonempty, convex and compact valued correspondence of a compact, convex set into itself (by Lemmas 8-9). By Kakutani's theorem it has a fixed point $(p^*, v^*, \theta^*, s^*)$, i.e.,

$$p^* \in P(\hat{u}(v^*)), \tag{9.2}$$

$$v^* \in V(p^*, \theta^*, s^*), \tag{9.3}$$

$$\theta^* \in \Theta(p^*, s^*), \tag{9.4}$$

$$s^* \in \hat{W}(\hat{u}(v^*)). \tag{9.5}$$

We now show that condition (ii) in the definition of a proportional solution is satisfied. By Lemma 4(i), (9.2) and (9.5) imply

$$p^* \cdot \tilde{z}(s^*) = 0, \tag{9.6}$$

that, recalling the definition of t^+ and t^-, can be rewritten as:

$$p^* \cdot (t^+(s^*) + t^-(s^*) - y_C^* - \omega_C) = 0 . \tag{9.7}$$

We claim that $\sigma_i(p^*, \theta^*, s^*) = 0$, i=1,...,M. By (9.1), $\sum_{i=1}^{M} \sigma_i(s^*, p^*, \theta^*) = 0$; hence, it suffices to show that $\sigma_i(p^*, \theta^*, s^*) \geq 0$ for i=1,...,M. Suppose not, i.e., $\sigma_i(p^*, \theta^*, s^*) < 0$ for some i. Then by (9.3), $\hat{u}_i(v^*) = 0 \leq U_i(\omega_i)$. Again by (9.2) and (9.5), Lemma 4 (i) implies that

$$p^* \cdot x_i^* \leq p^* \cdot \omega_i, \tag{9.8}$$

and, by Lemma 4 (ii) and Assumption A.3,

$$p^{*}\cdot y_g^{*} \geq 0, \; g=1,\dots,F, C. \tag{9.9}$$

or, combining (9.8) and (9.9),

$$p^{*}\cdot x_i^{*} \leq p^{*}\cdot\omega_i + \sum_{f=1}^{F} d_{if}\, p^{*}\cdot y_f^{*}. \tag{9.10}$$

By (9.7) and (9.9), $p^{*}\cdot t^{+}(s^{*}) + p^{*}\cdot t^{-}(s^{*}) \geq 0$, and because, by the definition of t^{+} and t^{-} we have that $p^{*}\cdot t^{+}(s^{*}) \geq 0$ and $-\,p^{*}\cdot t^{-}(s^{*}) \geq 0$, we obtain:

$$p^{*}\cdot t^{+}(s^{*}) = 0 \Rightarrow p^{*}\cdot t^{-}(s^{*}) = 0, \tag{9.11}$$

and

$$p^{*}\cdot t^{+}(s^{*}) > 0 \Rightarrow \frac{-p^{*}\cdot t^{-}(s^{*})}{p^{*}\cdot t^{+}(s^{*})} \in [0,\ 1]. \tag{9.12}$$

Consider first the case where $p^{*}\cdot t^{+}(s^{*}) = 0$. By Assumption A.1, $y^{*}_{fj} \leq 0$ for $j=K+1,\dots,N$, $f=1,\dots,F$, which, together with the facts that, by (9.11), $p^{*}\cdot t^{-}(s^{*}) = 0$, and that $p^{*}\cdot x_i^{*} \geq \sum_{j=1}^{K} p_j^{*} x_{ij}^{*}$, yields, using (9.10),

$$\sum_{j=1}^{K} p_j^{*} x_{ij}^{*} \leq p^{*}\cdot\omega_i + \sum_{f=1}^{F} d_{if} \sum_{j=1}^{K} p_j^{*} y_{fj}^{*} + \theta_i^{*} p^{*}\cdot t^{-}(s^{*}) \; ,$$

i.e, $\sigma_i(p^{*}, \theta^{*}, s^{*}) \geq 0$, contradicting the supposition that $\sigma_i(p^{*}, \theta^{*}, s^{*}) < 0$. Hence, $\sigma_i(p^{*}, \theta^{*}, s^{*}) = 0$, $i=1,\dots,M$, if $p^{*}\cdot t^{+}(s^{*}) = 0$.

Consider now the case where $p^{*}\cdot t^{+}(s^{*}) > 0$, and assume again that $\sigma_i(p^{*}, \theta^{*}, s^{*}) < 0$. We can rewrite (9.10) as:

$$\sum_{j=1}^{K} p_j^{*} x_{ij}^{*} \leq p^{*}\cdot\omega_i + \sum_{f=1}^{F} d_{if} \sum_{j=1}^{K} p_j^{*} y_{fj}^{*} - \left[\sum_{j=K+1}^{N} p_j^{*} x_{ij}^{*} - \sum_{f=1}^{F} d_{if} \sum_{j=K+1}^{N} p_j^{*} y_{fj}^{*}\right].$$

Again by Assumption A.1, the term in brackets is nonnegative and thus, using (9.12),

$$\sum_{j=1}^{K} p_j^{*} x_{ij}^{*} \leq p^{*}\cdot\omega_i + \sum_{f=1}^{F} d_{if} \sum_{j=1}^{K} p_j^{*} y_{fj}^{*} - \frac{-p^{*}\cdot t^{-}(s^{*})}{p^{*}\cdot t^{+}(s^{*})} \left[\sum_{j=K+1}^{N} p_j^{*} x_{ij}^{*} - \sum_{f=1}^{F} d_{if} \sum_{j=K+1}^{N} p_j^{*} y_{fj}^{*}\right]. \tag{9.13}$$

but, by (9.4) and because $p^*\cdot t^+(s^*) > 0$,

$$\theta_i^* = \frac{\sum_{j=K+1}^{N} p_j^* x_{ij}^* - \sum_{f=1}^{F} d_{if} \sum_{j=K+1}^{N} p_j^* y_{fj}^*}{p^*\cdot t^+(s^*)},$$

and (9.13) becomes

$$\sum_{j=1}^{K} p_j^* x_{ij}^* \leq p^*\cdot\omega_i + \sum_{f=1}^{F} d_{if} \sum_{j=1}^{K} p_j^* y_{fj}^* + \theta_i^* p^*\cdot t^-(s^*),$$

i.e., $\sigma_i(p^*, \theta^*, s^*) \geq 0$, contradicting the supposition that $\sigma_i(p^*, \theta^*, s^*) < 0$. Hence, $\sigma_i(p^*, \theta^*, s^*) = 0$, for $i=1,\ldots,M$. This together with the fact that, by (9.11), $p^*\cdot t^-(s^*) = 0$ whenever $p^*\cdot t^+(s^*) = 0$, yields condition (ii) in the definition of a proportional solution.

The proof will be completed after showing Pareto optimality. By the just proved condition (ii) and by Lemma 10, we have that for $i \in \{1,\ldots,M\}$,

$$p^*\cdot x_i^* = p^*\cdot\omega_i + \sum_{f=1}^{F} d_{if}\, p^*\cdot y_f^* + \theta_i(p^*,s^*)\, p^*\cdot[y_C^*+\omega_C], \qquad (9.14)$$

and because, by Lemma 4, profits are nonnegative, we have that for $i \in \{1,\ldots,M\}$, $p^*\cdot x_i^* \geq p^*\cdot\omega_i$. By Lemmas 4 and 5, $p^*\cdot x_h^* > 0$ for some h, and thus, by A.11 and Lemma 7, $p^*\cdot x_i^* > 0$ for $i \in \{1,\ldots,M\}$. Lemma 6 in turn guarantees utility maximization subject to a budget constraint where wealth is defined by the right hand side of (9.14). Pareto optimality now follows from the First Fundamental Theorem of Welfare Economics.

Proof of Theorem 6. Equation (9.6) above holds under the hypothesis of (A) or (B) and thus, by Lemma 10, the budget constraints in (A) and in the definition of a proportional solution are identical.

(A) Pareto optimality follows from the First Fundamental Theorem.

(B) The argument in the last paragraph of the proof of Theorem 5 shows profit maximization and utility maximization subject to $p^* \cdot x_i \leq p^* \cdot x_i^*$.

<u>Lemma 11</u>. <u>Let all consumers be identical and let</u> s* <u>be a symmetric equal benefit solution</u> (<u>i.e.</u>, $x_i^* = x_h^*$ <u>for all</u> i, h=1,...,M). <u>Then</u> s* <u>is a proportional solution</u>.

<u>Proof</u>. First, notice that $\theta_i(p^*,s^*) = 1/M$ (if $p^* \cdot t^+(s^*) > 0$, then $\theta_i(p^*,s^*) = \theta_h(p^*,s^*)$, i, h=1,...,M, and therefore, $\theta_i(p^*,s^*) = 1/M$, i=1,...,M). Second, by symmetry and feasibility, for j=1,...,N, $x_{ij}^* \leq \omega_{Ij} + \frac{1}{M}\sum_{f=1}^{M} y_{fj}^* + \frac{1}{M} y_{Cj}^* + \frac{1}{M}\omega_{Cj}$, whereas by (c) in the definition of a vector of efficiency prices, $p^* \cdot x_i^* = \frac{1}{M} p^* \cdot [M\omega_I + \sum_{f=1}^{F} y_f^* + y_C^* + \omega_C]$.

Thus, $p_j^* x_{ij}^* = p_j^* [\omega_{Ij} + \frac{1}{M}\sum_{f=1}^{F} y_{fj}^* + \frac{1}{M} y_{Cj}^* + \omega_{Cj}]$, j=1,...,N,

i.e., $\sum_{j=1}^{K} p_j^* x_{ij}^* = p^* \cdot \omega_I + \sum_{f=1}^{F} \frac{1}{M} \sum_{j=1}^{K} p_j^* y_{fj}^* + \theta_i(p^*,s^*) p^* \cdot t^-(s^*)$.

<u>Proof of Theorem 7</u>. By Theorem 4, a symmetric equal benefit solution exists and hence, by Lemma 11, it is a symmetric proportional solution.

<u>Proof of Theorem 8</u>. Because s^* is an equal benefit solution, for all h ε {1,...,M} we have that:

$$p^* \cdot x_h^* - p^* \cdot \omega_I - \frac{1}{M}\sum_{f=1}^{F} p^* \cdot y_f^* = \frac{1}{M}\left[p^* \cdot \sum_{\ell=1}^{M} x_\ell^* - Mp^* \cdot \omega_I - \sum_{f=1}^{F} p^* \cdot y_f^*\right]$$

i.e,

$$p^* \cdot x_h^* = \frac{1}{M} p^* \cdot \sum_{\ell} x_\ell^* . \tag{9.15}$$

and moreover,

$$p^* \cdot \left(\sum_{h=1}^{M} x_h^* - M\omega_I - \sum_{f=1}^{F} y_f^* - y_C^* - \omega_C\right) = 0,$$

which, together with feasibility, yields:

$$p_j^*\left(\sum_{h=1}^{M} x_{hj}^* - M\omega_{Ij} - \sum_{f=1}^{F} y_{fj}^* - y_{Cj}^* - \omega_{Cj}\right) = 0, \; j \in \{1,\ldots,N\}. \qquad (9.16)$$

<u>CASE 1</u>: $p^* \cdot x_h^* = 0$ for some h.

By (9.15), $p^* \cdot x_i^* = 0$, which in particular implies that:

$$p_j^* x_{ij}^* = 0 \quad , i=1,\ldots,M, \; j=1,\ldots,N, \qquad (9.17)$$

which, by (9.16) yields:

$$\sum_{j=1}^{K} p_j^*\omega_{Ij} + \frac{1}{M}\sum_{f=1}^{F}\sum_{j=1}^{K} p_j^* y_{fj}^* + \frac{1}{M}\sum_{j=1}^{K} p_j^*\left(y_{Cj}^* + \omega_{Cj}\right) = 0. \qquad (9.18)$$

By (9.17), $\theta_i(p^*,s^*) = \theta_h(p^*,s^*)$, i, h $\in$ {1,....,M}, and hence $\theta_i(p^*,s^*) = \frac{1}{M}$, for all i $\in$ {1,....,M}. Moreover, by (9.16) if $p_j^* t_j^-(s^*) < 0$ it must be the case that $p_j^* t_j^-(s^*) = p_j^*(y_{Cj}^* + \omega_{Cj}^*)$ and that either $\omega_{Ij} > 0$ or $y_{fj} > 0$ for some f $\in$ {1,....,F}, i.e., j $\in$ {1,....,K}. Thus, $p^* \cdot t^-(s^*) = \sum_{j=1}^{K} p_j^*(y_{Cj}^* + \omega_{Cj})$. Therefore, (9.18) implies that the right hand side of condition (ii) in the definition of a proportional equilibrium is zero. Because (9.17) implies that the left hand side is also zero, it follows that such a condition is satisfied.

<u>CASE 2</u>: $p^* \cdot x_h^* > 0$ for h=1,....,M.

Define the allocation $\hat{s}$ by keeping $\hat{y}_g = y_g^*$, g=1,....,F,C, and by setting all consumption vectors equal to $\hat{x} = \sum_{h=1}^{M} x_h^*/M$. Because $0 \in X_h$, x_h^* does not minimize

$p^* \cdot x_h$ on X_h and therefore, by standard arguments (see, e.g., Debreu 1959, (1) of 4.9, p. 69, or Arrow-Hahn, 1971, Lemma 4.3, p. 81) x^*_h maximizes $\hat{U}$ subject to $p^* \cdot x_h \leq p^* \cdot x^*_h$. Thus, by (9.15), $\hat{U}(x^*_h) = \hat{U}(x^*_i)$ for all h, i, and hence, by the concavity of $\hat{U}$, $\hat{U}(\hat{x}) \geq \hat{U}(x^*_h)$, h=1,...,M. But because s^* is Pareto optimal and $\hat{s}$ is feasible, we must have that $\hat{U}(\hat{x}) = \hat{U}(x^*_h)$, h=1,...,M, which in particular implies that $\hat{s}$ is Pareto optimal. It is now easy to check that $\hat{s}$ is a symmetric equal benefit solution and, hence, by Lemma 11, $\hat{s}$ is a proportional solution.

Before proving Theorem 9, we must formulate the proper version of FALE for a domain of economic environments with several inputs. Suppose there are two inputs, x_1 and x_2, privately owned, and the endowment of the i^{th} person is $(\omega_{11}, \omega_{12})$. Denote an environment $\xi = (\omega,U,f)$, where $f(x_1,x_2) = x_3$ and $\omega = (\omega_{11},\omega_{12},\omega_{21},\omega_{22},...,\omega_{M1},\omega_{M2})$. Consider the class of linear production functions $\Phi = \{f \mid \text{for some } \alpha, \beta \geq 0,\ f(x_1,x_2) = \alpha x_1 + \beta x_2\}$. Let $\sum^{\Phi}(\omega,U) = \{\xi \varepsilon \sum(\omega,U) \mid f \varepsilon \Phi\}$. If several persons work simultaneously on a technology in Φ, they produce the sum of what they produce separately for the same factor inputs. This is true only for $f\varepsilon\Phi$. Thus, the appropriate domain for the free access axiom is $\sum^{\Phi}$.

<u>Free Access in Linear Economies</u> (FALE): If F is an allocation mechanism defined on $\xi\varepsilon\sum^{\Phi}\omega,U)$ then $F(\xi)$ is the autarkic allocation.

<u>Proof of Theorem 9</u>. For any (ω,U), Lemma 1 continues to hold on $\sum(\omega,U)$, with the same proof. But F fails, in general, to be a montone utility path mechanism on most subdomains $\sum^{\Phi}(\omega,U)$ under FALE, and so no allocation mechanism satisfies PO, TMON, and FALE on $\sum_2$, the class of environments with two private goods. Consider a two person environment, and let $f(x_1,x_2) = \alpha x_1 + \beta x_2$. Choose another pair of coefficients (γ,δ) which does not dominate (α,β) nor is dominated by (α,β), a profile $U = (U_1,U_2)$ and endowments ω such that, under autarky, the utility of the first person

increases in moving from f to $g(x_1, x_2) = \gamma x_1 + \delta x_2$ and the utility of the second person decreases. This is always possible. It follows that on $\sum(\omega, U)$ F is not a montone utility path mechanism.

REFERENCES

Arrow, K. J.. (1969). "The Organization of Economic Activity: Issues Pertinent to the Choice of Market versus Non-Market allocation," in The Analysis and Evaluation of Public Expenditures: THE PPB System, p. 47-64. Joint Economic Committee of the Congress of the United States, Washington. D.C., 1969.

Arrow, K. J.. and F. H. Hahn, (1971), General Competitive Analysis. San Francisco: Holden Day.

Clarke, F. H.. (1983), Optimization and Nonsmooth Analysis. New York: Wiley.

Cohen, G. A., (1986), "Self Ownership. World Ownership and Equality. Part II. Social Philosophy and Policy 3, 76-96.

Cornet, B., (1987), "The Second Welfare Theorem in Nonconvex Economies." EUI Working Paper No. 87/265. European Economics Institute, Florence.

Debreu, G., (1979) Theory of Value. New York: Wiley.

Guesnerie. R., (1975), "Pareto Optimality in Non-Convex Economies," Econometrica 43(1), 1-29.

Israelsen, L. D., (1980), "Collectives, Communes and Incentives," Journal of Comparative Economics 4(2).99-124.

Kang, Suk. (1987). "Fair Distribution Rule in a Cooperative Enterprise," Journal of Comparative Economics, forthcoming.

Lindahl, E., (1919), Positive Losung. Die Gerchtigkeit der Besteurung. Lund. English translation in Musgrave and Peacock. Eds. (1985) Classics in the Theory of Finance.

Mas-Colell, A., (1980), "Remarks on the Game Theoretic Analysis of a Simple Distribution of Surplus Problem." International Journal of Game Theory 9, 125-140.

Mas-Colell, A., and J. Sitvestre, (1989), "Cost Share Equilibria: A Lindahlian Approach," Journal of Economic Theory. 47(2), 239-256.

Moulin, H., (1987), "A Core Selection for Pricing a Single Output Monopoly." Rand Journal of Economics 18(3), 397-407.

Moulin, H. and J. Roemer. (In press), "Public Ownership of the External World and Private Ownership of Self," Journal of Political Economy.

Newman. P.. (1987), "Consumption Sets," in J. Eatwell. M Milgate and P. Newman, Eds., The New Palgrave. London: MacMillan.

Roemer. J.. (1986). "The Mismarriage of Bargaining Theory and Distributive Justice," Ethics 97. 88-110.

Roemer. J., (1989), "A Public Ownership Resolution of the Tragedy of the Commons," Social Philosophy and Policy. 6. 74-92.

Sen, A. K. (1966), "Labour Allocation in a Cooperative Enterprise," Review of Economic Studies 33(4), 361-371.

Weitzman. M., (1974). "Free Access vs. Private Ownership as Alternative Systems for Managing Common Property," Journal of Economic Theory 8(2), 225-234.