*Uncovered Sets and Sophisticated Voting Outcomes with Implications for Agenda Institutions**

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This paper examines the properties of majority-rule institutions given fully strategic behavior by all agents. Results are provided, characterizing majority-rule outcomes, for several alternative agenda institutions. The main conclusion is that institutional arrangements, specifically mechanisms of agenda construction, impose constraints on majority outcomes.

In the last decade multidimensional voting models have become subtle and complex instruments for explicating social choices by majority rule. What has been learned from them is that little will be known about an institution based on majority rule if the focus is exclusively upon the majority preference relation between alternatives. Recent results in the theory of pure majority rule establish the generic character of majority preference cycles. Based on the theorems of McKelvey (1976, 1979), Cohen (1979), and Schofield (1978), the following assertions may be taken as characteristic of pure majority rule, given a modest diversity in individual preferences:

1. For any alternative, there is a nonempty set of alternatives, each element of which commands a majority against it.
2. The absence of a "majority undominated" alternative (Condorcet winner) is not an aberration, but rather the general case.
3. In multidimensional choice settings it is almost always possible to find a finite sequence of alternatives, beginning and ending with any arbitrary points, with each alternative in the sequence majority preferred to its predecessor.

Given the general absence of unbeaten alternatives, and the consequent prospect that, based on the majority preference relation, majority rule may "wander anywhere," McKelvey (1976, p. 480) was driven to the

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conclusion that equilibrium outcomes (if they exist) were "strongly dependent on the nature of the institutional mechanisms which generate the agenda." If a distinguished agent—the setter, convenor, proposer, chairman—has complete and exclusive power to choose and order the elements of an agenda, he may extract all of the advantage from a situation by selecting an appropriate sequence of votes to yield his own ideal point.

Each of the above conclusions depends upon a rather special sort of preference revelation. Individual agents are assumed to be "sincere" revealers of their preferences so that the majority preference relation (built up from sincerely revealed individual preferences) may be taken as descriptive of the voting behavior of majorities. At most, one agent—the agenda setter—is a strategic player. There is now, however, a developing noncooperative view of strategic behavior in voting situations (Farquharson, 1969; Kramer, 1972; McKelvey and Niemi, 1978; Enelow and Koehler, 1980; Miller, 1980; McKelvey, 1982a). The insights of these latter approaches have yet to be applied to a game-theoretic development of agenda institutions. This leads us to ask: Given fully strategic behavior by all agents, what are the operating characteristics and equilibrium states of alternative agenda institutions? In particular, we are interested in two endogenous agenda institutions—the centralized-agenda process, in which a distinguished agent has monopoly power to select and order alternatives (McKelvey, 1976; Romer and Rosenthal, 1978), and the open-agenda process, in which any agent may move an alternative (Ferejohn, Fiorina, McKelvey, 1981). But our results apply to many richer and more complex institutions as well, a point to which we return in the concluding section.

In the first two sections, we develop the idea of strategic or "sophisticated" (Farquharson, 1969) voting in an institutional context, and develop a constructive procedure for characterizing equilibrium outcomes. Section 3 contains our main result (Theorem 3) on the set of possible sophisticated outcomes of any agenda that commences at a predetermined point, y. It also has implications for an optimizing agenda setter in an environment of strategic agents. Section 4 gives a fuller characterization of this feasible set, known as the "uncovered set," relates it to the set of Pareto optimal alternatives, and provides several illustrative examples. A concluding discussion section summarizes the results and explores their implications for alternative agenda institutions.

1. Theoretical Preliminaries

We study agenda power in the context of multidimensional voting models of majority rule. We consider the standard setting in this literature in which a finite committee or legislature consisting of n agents, $N = \{1, \ldots, n\}$, must select a single element from a convex policy space $X \subset R^m$. We assume n is odd. The preferences of each agent are given by

a weak ordering $R_i \subset X \times X$. Thus $R_i(x) = \{y \in X \mid yR_ix\}$. P_i and I_i are the asymmetric and symmetric components of R_i, respectively. Thus, $P_i(x)$ and $I_i(x)$ are the interior and boundary, respectively, of $R_i(x)$. The former is i's preferred-to-x set and the latter is his indifferent-to-x set. Throughout we employ the following regularity assumptions common to multidimensional voting models:

A.1. Continuity. For all $i \in N$, R_i is continuous. Hence $R_i(x)$ and $R_i^{-1}(x)$ are closed.

A.2. Strict quasi-concavity. For all $i \in N$ and $x, y \in X$, if $y \in R_i(x)$ and $z = ty + (1-t)x$, $0 < t < 1$, then $z \in P_i(x)$.[1]

A.3. Compact preferences. For all $i \in N$ and $x, y \in X$, $R_i(x)$ is compact.

In addition, we define, for each $i \in N$, a distinguished point $x^i \in X$, called i's ideal point, with the property that $x^iP_iy \ \forall y \neq x^i$.

Winning coalitions are defined in terms of (relative) majority rule. The majority-rule preference relation $P \subset X \times X$ is given by

$$xPy \text{ iff } |\{i \mid xP_iy\}| > |\{i \mid yP_ix\}|.$$

Thus, x is majority preferred to y if and only if the number of agents strictly preferring x to y exceeds the number strictly preferring y to x. R and I are defined analogously (with ">" replaced by "≥" and "=," respectively, in the definition of P).

Finally, we define several correspondences based on P. The win set of x, $W(x)$, is the set of points majority preferred to x:

$$W(x) = \{y \mid yPx\}.$$

Its inverse,

$$W^{-1}(x) = \{y \mid xPy\},$$

is the set of points to which x is majority preferred. Analogously, the closure and boundary of W are given by

$$R(x) = \{y \mid yRx\} \text{ and } I(x) = \{y \mid yIx\},$$

respectively. Under the stated assumptions[2] it may be established that

1. $P_i(x)$ is open and $I_i(x)$ is closed with no interior, that is, "thin indifference" sets (McKelvey, 1979, Lemma 2a);

[1] Hence individual indifference sets are "thin," viz., the interior of $I_i(x)$ is empty for all $i \in N$ and all $x \in X$.

[2] If n is even then strictly quasi-concave preferences do not yield, under majority rule, "thin tie sets." With tie regions, i.e., $I(x)$ sets with nonempty interiors, some of our results either no longer hold in the strong form in which they are stated or require much more tedious development to establish. Here we develop the stronger results under the "n odd" stricture. The issue of ties is addressed again shortly.

2. $W(x)$ and $W^{-1}(x)$ are open (McKelvey, 1979, Lemma 3a); and
3. $I(x)$ is closed with no interior, that is, "thin tie" sets (McKelvey, 1979, Lemma 3a).

This last fact allows us to state, without proof, a result of which we make considerable use:

> *Lemma T (Thin Tie Sets).* Let $y \epsilon I(x)$. Then for any open neighborhood of y, $N(y)$,
> (i) $N(y) \cap W(x) \neq \phi$
> (ii) $N(y) \cap W^{-1}(x) \neq \phi$.

In words, if y ties x, then any neighborhood containing y contains points that beat x and points that lose to x. Figure 1 illustrates these properties.

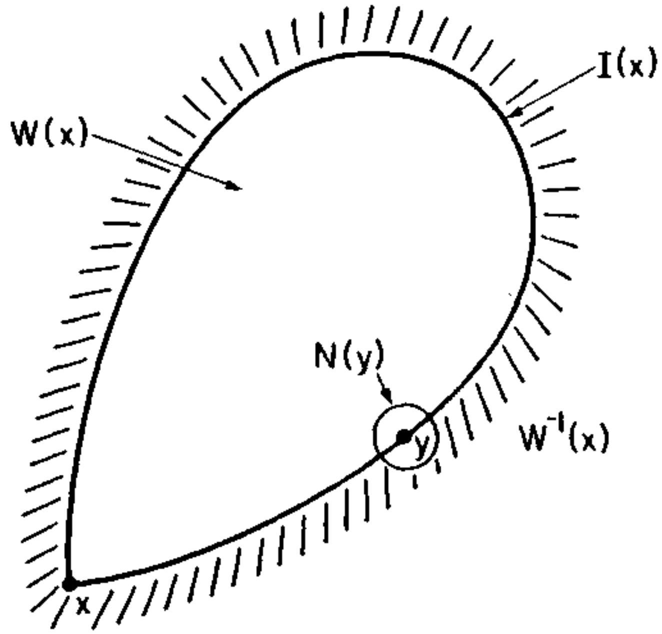

FIGURE 1

Properties of Thin Tie Sets (Lemma T)

Decision making proceeds according to an agenda $V = (x_1, \ldots, x_n)$, a finite, ordered subset of X. Once V is given,[3] voting commences as a majority-rule, sequential-elimination process, known as the amendment procedure. The first pair of agenda alternatives is voted upon at the first division with the one obtaining the greater number of votes advancing to the next division. There it is paired against the next alternative and the process is repeated. The ultimate surviving alternative is declared the winner of agenda V. If, at any division, each alternative receives the same number of votes (which is possible if some individuals are indifferent), then our convention for tie breaking is to advance the alternative listed *later* in the agenda to the next division. We call this the successor rule for ties.[4]

2. Sophisticated Voting Behavior

We begin our analysis of sophisticated behavior in majority-rule institutions with an example consisting of six alternatives. Consider an agenda $V = (x_1, x_2, x_3, x_4, x_5, x_6)$, which is displayed in the division scheme of Figure 2. The majority preference relation, P, is given in Figure 3 for some specific legislature. An entry of "1" in a cell means the row alternative is preferred by a majority to the column alternative; "0" implies the opposite. P is seen to be complete, asymmetric, and cyclic in this particular example. That is, there is no Condorcet winner $[W(x) \neq \phi \ \forall \ x \epsilon V]$. To determine the social choice in this example, we must make assumptions about individual voting behavior. We consider two possibilities:

> (i) *Sincere voting.* At each division in the tree given in Figure 2 each legislator votes for the alternative in the pair that stands highest in his preference ordering; sincere voting is nonstrategic.
> (ii) *Multistage sophisticated (MS) voting.* The division scheme of Figure 2 is treated as a multistage game with five levels. At the last level—level 5—one of 16 choice situations confronts legislators. At this final division, each votes sincerely as there are no incentives to vote strategically at these ultimate divisions. Consequently for

[3] The agenda may be constructed in either of two ways. We shall say that the agenda is constructed *forward* if the order in which alternatives are voted is precisely the order in which they were initially moved (or otherwise placed on the agenda.) Contrariwise, an agenda is constructed *backward* if alternatives are voted on in precisely the reverse order in which they are moved.

[4] In the literature we have identified several arbitrary devices for handling problems of ties. We have used what we believe is the least restrictive in which the resolution of ties depends upon the order in which alternatives are voted on. Alternative restrictions on ties include: (1) strict agenda (which defines an admissible agenda as one over whose elements "winning" is well-defined), e.g., McKelvey (1982a); (2) assuming no ties in underlying set of alternatives as in a strict tournament, e.g., Miller (1980); or (3) a strong, proper, simple game, e.g., McKelvey (1982b). In addition, there have been devices by which nonstrong games are made strong, e.g., a nonanonymous procedure identifying an individual, or strict hierarchy of individuals, with the power to break ties.

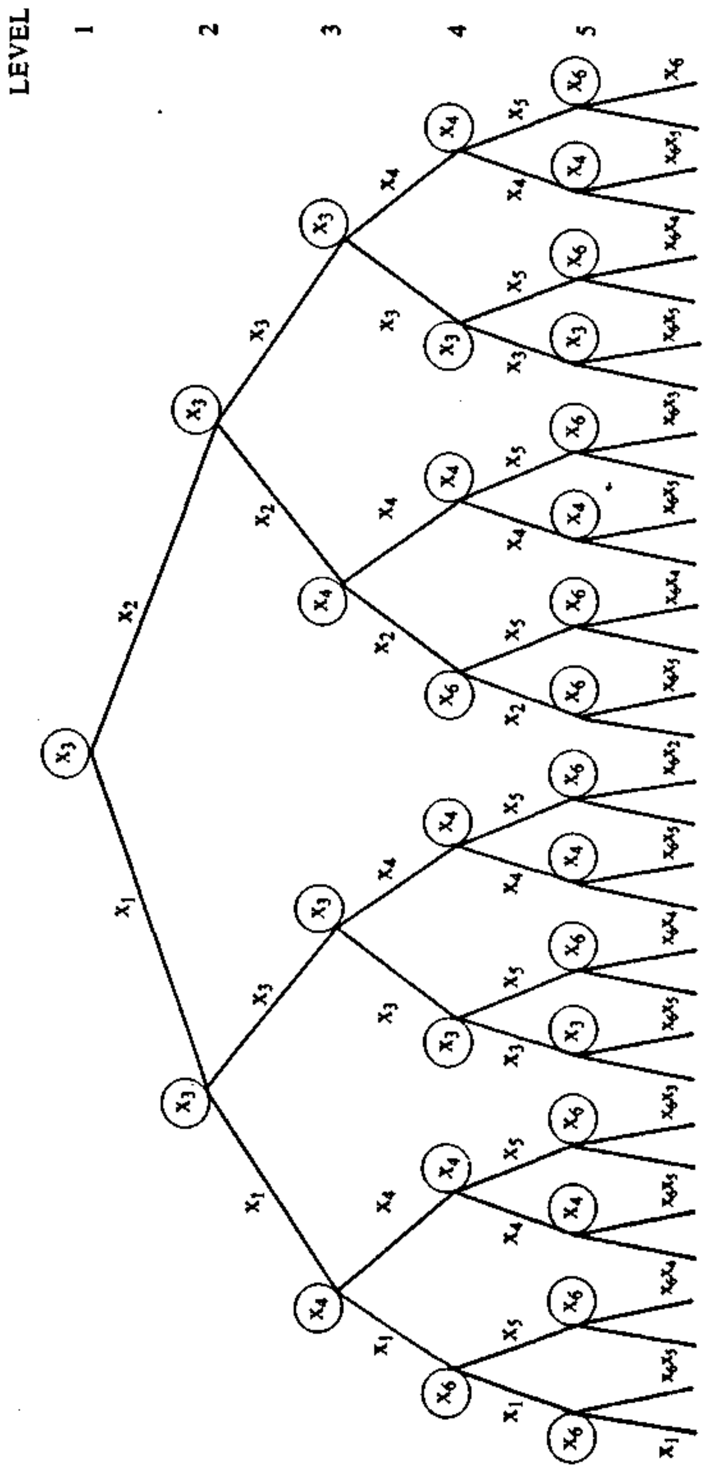

FIGURE 2
Division Scheme for Multistage Game with Five Levels

FIGURE 3
Majority Preference Relation for $V = (x_1, x_2, x_3, x_4, x_5, x_6)$

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	—	1	1	0	0	0
x_2	0	—	1	0	0	0
x_3	0	0	—	1	0	1
x_4	1	1	0	—	1	1
x_5	1	1	1	0	—	0
x_6	1	1	0	0	1	—

this division the majority preference relation of Figure 3 describes choice behavior as well as preferences. For each of the 16 level-5 nodes, the majority winner becomes the sophisticated equivalent of its node (McKelvey and Niemi, 1978). The level-4 nodes now represent choices between sophisticated equivalents—between, that is, what will happen at level 5 if a particular choice is made at level 4. The majority preference relation applied to sophisticated equivalents (in contrast to the nominal alternatives actually being voted on) describes strategic behavior. Thus we may again attach a sophisticated equivalent to each level-4 node. By repeating this algorithm we may work our way up the division scheme with the sophisticated equivalent attached to the single level-1 node, the sophisticated outcome. In Figure 2, sophisticated equivalents are attached in circles to each node.[5]

We may now discuss each of the behavioral hypotheses as they apply to our example of Figures 2 and 3. Consulting the majority preference relation of Figure 3, the outcome of sincere voting is easily determined

[5] MS voting is an intuitively more satisfactory rendering of strategic behavior in structured (decision tree) settings than the original notion of sophisticated voting presented by Farquharson (1969). The Farquharson procedure, based on a binary decision tree like Figure 2, is described as follows:

> For m alternatives there are $k = 2^{m-1} - 1$ divisions under the amendment procedure. At each division, a voter votes either left (L) or right (R) so that there are 2^k strategies available to him, where a strategy is a k-tuple of Ls and Rs. With n legislators, there are then $(2^k)^n$ possible plays of the amendment procedure game. The Farquharson procedure is that of dominance elimination. Each legislator eliminates dominated strategies and assumes each other legislator does the same. This produces a reduced game in which each legislator again eliminates all dominated strategies and again assumes each other legislator does the same. This reduction procedure is repeated until no further strategy elimination is possible; each legislator is now left with his "sophisticated strategies," and the outcomes are "sophisticated equilibria."

Gretlein (1980) shows that the Farquharson procedure and the MS procedure yield identical outcomes, so we do not separately pursue the former.

(with or without direct reference to the division scheme): x_1 defeats x_2; x_1 defeats x_3; x_1 loses to x_4; x_4 defeats x_5; and x_4 defeats x_6. Sincere voting produces x_4 as the outcome. A sophisticated agent, on the other hand, determines his own strategy as well as the sophisticated choices of majorities at each node by deducing what will occur if the process reaches that node. Beginning at the 16 level-5 nodes, the element circled—the sophisticated equivalent—is the alternative that P-dominates if that node is reached. The level-4 decisions, then, are really choices between the two sophisticated equivalents at level 5 for each level-4 node. Applying P to that choice gives us the circled level-4 sophisticated equivalents. Moving up the tree in this fashion, we see that x_3 is the MS outcome.

The MS formulation exploits a recursive logic applied to finite game trees. But it fails to exploit geometric and set-theoretic intuitions when alternatives are given a richer geometric or topological structure. Our own approach to be formalized below incorporates this structure to provide a more intuitive and simpler procedure. It is indicated by returning to Figure 3. Begin with the last alternative in the agenda, x_6. Notice that any alternative, if it is to be the final outcome, must prevail over x_6 in the final division. Thus those alternatives that lose to x_6—x_1, x_2, x_5—cannot be the sophisticated outcome. Only x_3 and x_4 defeat x_6 and between them x_3 defeats x_4. We shall show that this implies x_3 is the sophisticated outcome.

This process for computing the sophisticated outcome can be formalized usefully as follows. Consider an agenda $V = (x_1, \ldots, x_n)$. Characterize an alternative $x_i \epsilon V$ as *innocuous* if it is unable to defeat each noninnocuous motion succeeding it in the voting order. The notion of innocuousness permits us to construct the *sophisticated agenda* Z = $(z_1, \ldots, z_n)$ of noninnocuous motions from V:

$$z_n = x_n \qquad \text{since } x_n \text{ is trivially noninnocuous;}$$

$$z_{n-1} = \begin{cases} x_{n-1} & \text{if } x_{n-1} \in W(x_n); \\ z_n & \text{otherwise,} \end{cases}$$

$$z_{n-2} = \begin{cases} x_{n-2} & \text{if } x_{n-2} \in W(z_n) \cap W(z_{n-1}); \\ z_{n-1} & \text{otherwise} \end{cases}$$

$$\vdots$$

$$z_1 = \begin{cases} x_1 & \text{if } x_1 \in \bigcap_{i=2}^{n} W(z_i) \\ z_2 & \text{otherwise} \end{cases}$$

Generally, $z_i = x_i$ *iff* $x_i \in \bigcap_{j=i+1}^{n} W(z_j)$; otherwise $z_i = z_{i+1}$.

In effect, Z is a subordering of V, assembled an element at a time

beginning with the last element of V and working backwards. Thus in our example:

$$z_6 = x_6 \text{ since } x_6 \text{ is trivially noninnocuous;}$$

$$z_5 = x_6 \text{ since } x_5 \notin W(x_6);$$

$$z_4 = x_4 \text{ since } x_4 \in W(x_6);$$

$$z_3 = x_3 \text{ since } x_3 \in W(x_4) \cap W(x_6);$$

$$z_2 = x_3 \text{ since } x_2 \notin W(x_4) \cap W(x_6);$$

$$z_1 = x_3 \text{ since } x_1 \notin W(x_4) \cap W(x_6).$$

Notice that since $x_5 \notin W(x_6)$ and hence is innocuous, it does not matter that $x_3 \notin W(x_5)$. Moreover, since neither x_1 nor x_2 is in the appropriate intersection, they, too, are innocuous and are deleted.

For the agenda and P-relation of Figures 2 and 3, the sophisticated agenda according to this procedure is $Z = (x_3, x_3, x_3, x_4, x_6, x_6)$. Whenever it causes no confusion, we truncate by writing $Z = (x_3, x_4, x_6)$.[6]

Throughout the rest of this paper, we concentrate on voters who are strategic agents, those who vote sophisticatedly. We may now state two properties of our procedure.

Theorem 1 (Equivalence Theorem): For a given agenda, V, the first element of the sophisticated agenda is identical to the sophisticated outcome of the MS formulation.
Proof. See Appendix.

Throughout we refer to z_1, the first element of Z, as the sophisticated agenda equilibrium (sae) of the agenda V. It identifies the equilibrium outcome associated with any V when all agents vote sophisticatedly. We now have the mapping that relates the class of finite, sequenced agendas to specific outcomes. By virtue of the method of constructing the sophisticated agenda, we may state as a theorem the defining property of the sae of any V:

Theorem 2 (Intersecting-Win-Sets Theorem). For a given agenda V of length k, and the associated sophisticated agenda Z, the sae, $z_1 \epsilon Z$, satisfies

$$z_1 \in \bigcap_{j=2}^{k} W(z_j).$$

[6] The algorithm that gives Z as a vector of same length as V simplifies the proof of the equivalence theorem in the Appendix. In most examples, we will write Z in truncated form.

Thus we have exhibited a new approach to sophisticated voting over finite, ordered agendas. Our theorem gives a constructive method (the sophisticated agenda) of finding the sophisticated voting outcome of any agenda under the amendment process. It is much simpler than the MS formulation since the latter involves one calculation (of the sophisticated alternative) for each node while ours requires only one calculation per level. Theorem 1 insures that these procedures arrive at the same answer.[7]

3. Majority-Rule Outcomes under Sophisticated Behavior

Having developed our theory of sophisticated behavior, we now apply this approach to questions relevant to majority rule. Our first model of agenda setting is that of McKelvey (1976), except that all agents are sophisticated. Since we are interested in contrasting our results with those of McKelvey, we address the same question he posed. Commencing at a given point $y \epsilon X$, what outcomes can be produced with an appropriately chosen agenda? This is, building an agenda forward (see note 3), what is the range of sophisticated outcomes derived from the domain of finite agendas commencing at y?

These issues have been extensively explored for the case of sincere voting. McKelvey (1976, 1979) proves that there exists a finite agenda that yields any point in X as the majority-rule outcome when there is no Condorcet winner (see below for a formal statement of his result). No results have been published to date concerning the sophisticated case. Moreover, we know little about the power of an agenda setter in the context of sophisticated voting and a multidimensional alternative space, although there are some conjectures by Miller (1980). The thrust of Miller's analysis is that, as McKelvey (1976) noted in qualification to his theorem, sophistication by other agents limits the influencne of a centralized agenda setter. We now take up these issues.

Principal Results

We begin by defining the covering relation.

Definition. $\forall\ x, y \epsilon X$, y covers x (yCx) if and only if

(i) $y \epsilon W(x)$

(ii) $W(y) \subset W(x)$.

The point x is covered by y if yPx and if every alternative that is majority preferred to y also is majority preferred to x.[8] Miller (1980) should be credited with first calling attention to the uncovered set.[9] The covering relation between alternatives bears a close relationship to the game-theoretic dominance relation between strategies (McKelvey and Ordeshook, 1976).[10] From the covering relation, several preliminary lemmas may be established.

Lemma 1. $W(y) \subset W(x) \rightarrow W^{-1}(x) \subset W^{-1}(y)$

Proof. Suppose the contrary: $W(y) \subset W(x)$ but $z \epsilon W^{-1}(x)$ and $z \notin W^{-1}(y)$. Then $z \epsilon I(y)$ (since, if $z \epsilon W(y)$ then, by hypothesis, $z \epsilon W(x)$, a contradiction). $z \epsilon W^{-1}(x) \rightarrow \exists\ N(z) \subset W^{-1}(x)$ (since $W^{-1}(x)$ is open). But, from $z \epsilon I(y)$ and Lemma T, $\exists\ u \epsilon N(z)$ with $u \epsilon W(y)$. From the hypothesis, $u \epsilon W(x)$, contradicting $N(z) \subset W^{-1}(x)$.

Q.E.D.

Corollary 1. $yCx \rightarrow W^{-1}(x) \subset W^{-1}(y)$.

Proof. $yCx \rightarrow W(y) \subset W(x)$. Lemma 1 establishes the result.

Q.E.D.

Lemma 2. (a) $[y \epsilon W(x) \text{ and } \sim yCx] \rightarrow W(y) \cap W^{-1}(x) \neq \phi$.

(b) $W(y) \cap W^{-1}(x) \neq \phi \rightarrow \sim yCx$.

Proof. (a) From the definition of covering, the hypothesis implies $W(y) \not\subset W(x)$. Hence $\exists\ z \epsilon W(y)$ such that $z \notin W(x)$. If $z \epsilon W^{-1}(x)$, the result is established directly. So suppose $z \epsilon I(x)$. From $z \epsilon W(y)$, construct a neighborhood $N(z) \subset W(y)$ (since $W(y)$ an open set). From Lemma T, and $z \epsilon I(x)$, $N(z) \cap W^{-1}(x) \neq \phi$. Choose $u \epsilon N(z) \cap W^{-1}(x)$. But $u \epsilon W(y)$ since $N(z) \subset W(y)$. Thus $u \epsilon W(y) \cap W^{-1}(x)$ and the result follows.

(b) $z \epsilon W(y) \cap W^{-1}(x) \rightarrow z \epsilon W(y)$ and $z \notin W(x) \rightarrow W(y) \not\subset W(x)$. Thus $\sim yCx$.

Q.E.D.

We now state our main result.

Theorem 3. There exists a finite agenda, with y the first element and x the sae, if and only if $\sim yCx$.

[7] McKelvey and Niemi (1978) assume away the problem of ties by restricting their analysis to those agendas for which P is complete. We do not restrict agendas in this fashion. Instead we award victory, in case of a tie, to the "succeeding" element in the agenda. If we had chosen to use a predecessor tie-breaking rule instead, then the intersecting sets of Theorem 2 would have been the $R(z_i)$ sets.

[8] McKelvey (1982b) adds a third requirement to his definition of covering: $R(y) \subseteq R(x)$. From Lemma T and our definition, however, this requirement is implied: Suppose y covers x and $z \epsilon R(y)$. If $z \epsilon W(y)$, then $z \epsilon W(x)$, by (ii) of covering, and hence in $R(x)$. So suppose $z \epsilon I(y)$ and, contrary to hypothesis, $z \notin R(x)$. Thus $z \epsilon W^{-1}(x)$. Construct a neighborhood around z, $N(z)$, such that $N(z) \subset W^{-1}(x)$ (which is possible since $W^{-1}(x)$ is an open set). Thus every $u \epsilon N(z)$ satisfies $u \epsilon W^{-1}(x)$. But, by Lemma T, for some $u \epsilon N(z)$, $u \epsilon W(y)$. And by (ii) of covering $W(y) \subset W(x)$. So, we have $u \epsilon W(x)$. But we have already established $u \epsilon W^{-1}(x)$ for all u, a contradiction. Thus $R(y) \subseteq R(x)$. Q.E.D. In sum, thin tie sets imply that whenever $W(y)$ is contained in $W(x)$, their closures satisfy the same property.

[9] It is important to note that Miller defines covering relative to a fixed, finite, exogenously given agenda V. On the other hand, we define covering relative to the entire space in which any finite agenda is embedded.

[10] Specifically, y *dominates* x if $W(y) \subset W(x)$. For y to cover x, it is also required to beat x.

Proof. Sufficiency: Suppose $\sim yCx$. Then either $y \notin W(x)$ or $W(y) \not\subset W(x)$. If the former, then the agenda $V = (y, x)$ yields x since $x \in W(y)$ produces x as sae directly, and $x \in I(y)$ produces x as sae by the successor tie-breaking rule. If $W(y) \not\subset W(x)$ and $y \in W(x)$, then Lemma 2a imlies there exists a $z \in W(y) \cap W^{-1}(x)$. Then agenda $V = (y, x, z)$ yields x as sae since y is rendered innocuous by z.
Necessity: Suppose yCx and, contrary to hypothesis, there is an agenda V, $x,y \in V$, with x the sae. (i) If y precedes x on V, then x = sae(V) implies x is the first element of the sophisticated agenda, Z (Theorem 1). By Theorem 2, $x \in W(z)$ $\forall$ $z \in Z$, so that $z \in W^{-1}(x)$. But, from covering and Corollary 1 we have $W^{-1}(x) \subset W^{-1}(y)$ so that every element of sophisticated agenda is also beaten by y. Moreover, since yCx, we have $y \in W(x)$. Since y precedes x on V, x cannot be the first element of the sophisticated agenda. (ii) If x precedes y then y cannot be on the sophisticated agenda (if it were, then $x \notin W(y)$ would render x innocuous). This implies $\exists$ $z \in V$ with $z \in Z$ and $y \notin W(z)$. But yCx implies $W^{-1}(x) \subset W^{-1}(y)$ (Corollary 1) so that $x \notin W(z)$. Hence $x \neq$ sae(V). Contradiction.

Q.E.D.

This theorem demonstrates the power of covering and completely characterizes the set of feasible sae's when agents are sophisticated. So long as a point is not covered by a given commencement alternative, there exists some agenda that yields it as the consequence of sophisticated voting. Thus in the majority-rule context of sophisticated agents we have provided an answer to McKelvey's (1976) question: What alternatives may be achieved as the sae of some finite agenda V commencing at predetermined $y \in X$?

Contrast Theorem 3 with McKelvey's Theorem (1976, Theorem 2) for sincere voting:

McKelvey's Theorem: If there is no Condorcet winner,[11] then for any $x,y \in X$, there exists a finite agenda commencing at y and ending at x, $V = (y, z_1, \ldots, z_n, x)$, with $z_1 P y$, $z_i P z_{i-1}$, and $x P z_n$.

McKelvey's Theorem states that, from any initial point, there is an agenda that will lead *sincere* voters to any terminal point. By contrast, our theorem states that, from any initial point, there is an agenda that will lead *sophisticated* voters to any point not covered by the initial point. Clearly then, sophisticated agent behavior constrains majority rule relative

[11] The conditions required for a point to be a Condorcet winner (yPx $\forall$ $x \in X$) are given by Plott (1967), McKelvey (1976, 1979), Cohen (1979), and Schofield (1978). In the context of Euclidean preferences, they amount to requiring that a point be a total median—every hyperplane through the point must partition X into two halfspaces, the closures of which each contain at least half the voter ideal points.

to sincere behavior. Moreover, its effect is quite general and holds for any agenda-formation process. How binding the constraint is will be examined below.

There are two important consequences of Theorem 3, but several additional facts need first to be established. The first, stated without proof, follows from the starlike character of $W(x)$.[12]

Lemma 3. Under assumption A.2, for any $y \in X$ and neighborhood $N(y)$, if $W(y) \neq \phi$ then
(i) $N(y) \cap W(y) \neq \phi$
(ii) $N(y) \cap W^{-1}(y) \neq \phi$.

We use these facts momentarily after some further terms are defined.

Definition: For $i > 1$, the sets $W^i(x)$ and $W^{-i}(x)$ are given by
$W^i(x) = \{y \mid y \in W(z)$ and $z \in W^{i-1}(x)\}$ and
$W^{-i}(x) = \{y \mid y \in W^{-1}(z)$ and $z \in W^{-i+1}(x)\}$, respectively.

$W^i(x)$ is the set of points reachable from x, according to the P-relation, in i steps. Thus, $W^2(x) = \{y \mid y \in W(z)$ and $z \in W(x)\}$ is the set of points for which $yPzPx$ holds (for some z). The closures of $W^i(x)$ and $W^{-i}(x)$ ($R^i(x)$ and $R^{-i}(x)$, respectively) are defined analogously.

Lemma 4. $\forall$ $x \in X$, (i) $W(x) \subset W^2(x)$, and (*ii*) $W^{-1}(x) \subset W^{-2}(x)$.
Proof. Suppose $y \in W(x)$. Choose a neighborhood $N(y) \subset W(x)$ (possible since $W(x)$ is an open set). By Lemma 3 (ii), $W(x) \cap W^{-1}(y) \neq \phi$, so $\exists$ $z \in W(x)$ and $z \in W^{-1}(y)$. But this implies $y \in W^2(x)$ from the definition of $W^2(x)$.
(ii) Same proof as in (i).

Q.E.D.

Lemma 5. If $\sim yCx$ then $y \in R^{-2}(x)$. If x is uncovered (yCx for no $y \in X$) then $R^{-2}(x) = X$.
Proof. By definition, $W^{-2}(x) \subset R^{-2}(x)$. We show the somewhat stronger result that $\sim yCx$ implies $y \in W^{-2}(x)$ in all but one circumstance. Assume $\sim yCx$ and consider four cases:

(i) If $y \in W^{-1}(x)$ then $y \in W^{-2}(x)$ by Lemma 4(ii).
(ii) If $y \in W(x)$ then, from the hypothesis that $\sim yCx$, $W(y) \cap W^{-1}(x) \neq \phi$ by Lemma 2(a). Thus $\exists$ a $z \in W^{-1}(x)$ with $y \in W^{-1}(z)$, that is, $y \in W^{-2}(x)$.
(iii) If $y \in I(x)$ and $W(y) \not\subset W(x)$, then $\exists$ $z \in W(y)$ with $z \notin W(x)$. If $z \in W^{-1}(x)$ then $y \in W^{-2}(x)$ from the definition of the latter. If, on the other hand, $z \in I(x)$, then construct $N(z) \subset W(y)$. From Lemma T, $N(z) \cap W^{-1}(x) \neq \phi$. In particular, $\exists$ $u \in N(z) \cap W^{-1}(x)$ with $u \in W(y)$—since $N(z) \subset W(y)$—and $u \in W^{-1}(x)$. Hence $y \in W^{-1}(u)$ and $u \in W^{-1}(x)$, so $y \in W^{-2}(x)$.

[12] These results are proven in

(iv) If $y \epsilon I(x)$ and $W(y) \subset W(x)$, then from Lemma 1, $W^{-1}(x) \subset W^{-1}(y)$. So there is no $z \epsilon W^{-1}(x) \cap W(y)$ and hence y may not be an element of $W^{-2}(x)$. We therefore show that $y \epsilon R^{-2}(x)$. From Lemma T(ii) and the definition of $R^{-1}(x)$, $\exists$ $N(y)$ with $N(y) \cap R^{-1}(x) \neq \phi$. Pick $z \epsilon N(y) \cap R^{-1}(x)$ such that $z \epsilon I(y)$ (see Figure 4). Then $z \epsilon R(y)$ and $z \epsilon R^{-1}(x)$, so $y \epsilon R^{-2}(x)$.[13]

The second statement of the lemma follows directly.

Q.E.D.

We may now establish two corollaries of Theorem 3.

Corollary 3.1 (The Two-step Principle). Commencing at y, for any point that is the sae of some k-step agenda, there is an agenda with at most two steps possessing the same sae.

Proof. Let x be the sae of some k-step agenda beginning at y. By Theorem 3, $\sim yCx$. This implies three possibilities: (i) $y \epsilon W^{-1}(x)$,

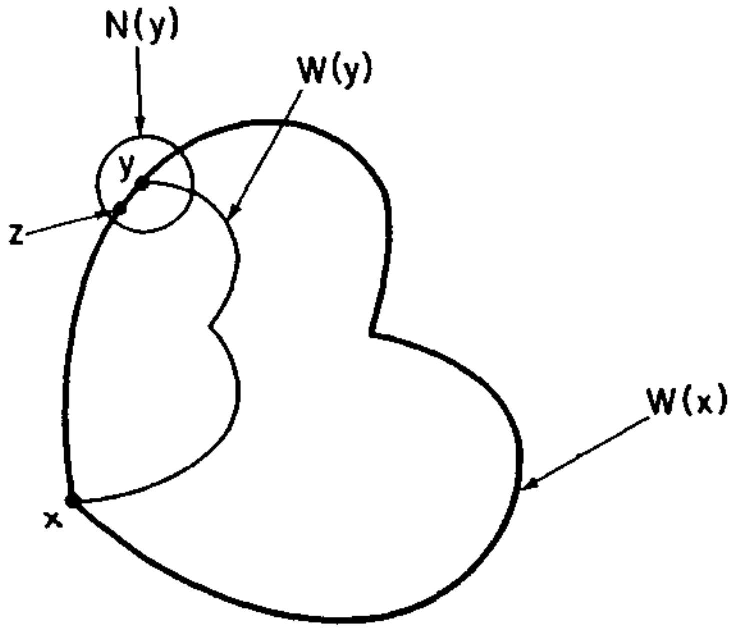

FIGURE 4

Possible Configuration of a Point Not Covered by x

[13] The circumstances of case (iv) pictured in Figure 4—$y \epsilon I(x)$ and $W(y) \subset W(x)$—permit only the weaker result that $\sim yCx$ implies $y \epsilon R^{-2}(x)$. In all other circumstances, the hypothesis yields $y \epsilon W^{-2}(x)$. We leave as open the question of whether, under the assumptions of our model (n odd, A.1–A.3, and relative majority rule), the circumstances of case (iv) can ever arise. If not, then the stronger result may be obtained.

(ii) $y \epsilon W(x)$, (iii) $y \epsilon I(x)$. If (i), then the agenda $V = (y, x)$ yields x as the sae in one step. If (ii), then Lemma 5 and the definition of covering imply $\exists$ $z \epsilon W(y) \cap W^{-1}(x)$; so the agenda $V = (y, x, z)$ yields x as the sae in two steps since z renders y an innocuous motion. If (iii), then the successor tie-breaking rule and the agenda $V = (y, x)$ give the desired result.

Q.E.D.

In short, if x is the sae for some agenda in which y is the commencement point, then it is possible to attain it in no more than two votes.[14]

The implication of these results for a centralized agenda setter is given, without proof, by

Corollary 3.2. If the ideal point of the agenda-setter, x^{AS}, is not covered by y, then there is an agenda commencing at y with x^{AS} as its sae.

These are rather remarkable results in two respects. First, contrary to the anticipation of Miller (1980), sophisticated agent behavior may not constrain an optimizing agenda setter to a small, centrally located subset. This will depend, of course, on what constitutes the set of points not covered by y (examined below). Second, if the agenda setter's ideal point, x^{AS}, is not covered by y, then there is at most a three-element agenda, with y in the first division, that produces x^{AS} as the sae.[15]

This set of results underscores the importance of the covering relation. First, any point in an agenda covered by some other point in

[14] The two-step principle has relevance for sincere voting, too. McKelvey's Theorem implies the existence of a finite agenda from y to x under sincere voting, but he does not specify its length. We can assert that if $\sim yCx$ then there is a two-step agenda that accomplishes the result: (i) if $x \epsilon W(y)$, then $V = (y, x)$ gives x as the sincere outcome; (ii) if $x \epsilon I(y)$ then $V = (y, x)$ and the successor tie-breaking rule accomplishes the same result; finally, (iii) if $y \epsilon W(x)$, then $\sim yCx$ implies some $z \epsilon X$ with $z \epsilon W(y) \cap W^{-1}(x)$ (Lemma 2.a); the agenda $V = (y, z, x)$ gives x as the sincere outcome. In sum, with sophisticated agents there is a two-step agenda from y to any point not covered by y (and only to those points). With sincere agents there is a two-step agenda from y to any point not covered by y, and a finite (more than two-step) agenda from y to any other point.

[15] In a personal communication, Nicholas Miller provided us with an extremely interesting conjecture. Miller's examples show that, for a $y \epsilon X$ centrally located relative to agent ideal points, the set of points not covered by y shrinks, relative to the distribution of ideal points, as the number of agents increases. For a sufficiently large number of agents, the set of points not covered by y is contained entirely within the convex polyhedron of their ideal points. In terms of our results, this conjecture suggests that strategic behavior in small committees and large legislatures may have different consequences. In particular, in the latter, independent strategic behavior by agents may be sufficient to constrain the social choice process while, in the former, strategic behavior may be only mildly constraining. While we do not pursue this question here, it clearly lies at the heart of an important normative issue: When is individual adaptive behavior sufficient to protect agents from exploitation by those with agenda power and, on the other hand, when are additional structural safeguards necessary?

that agenda cannot be the sae. Second, starting at a predetermined point, there is a two-step agenda yielding as its sae any point not covered by the predetermined commencement point. Furthermore, the results in this section are quite general; they apply to any agenda-formation process.

4. Characterizing Uncovered Sets

The covering relation plays a central role in sophisticated voting, as the previous theorems demonstrate. Theorem 3, in particular, focuses attention on

$$UC(y) = \{x \mid \sim yCx\},$$

the correspondence that maps an element of X to the set of elements of X not covered by it (the latter elements, according to Theorem 3, are available as potential sae's of some agenda commencing with the particular domain point of UC). The purpose of the examples and propositions that follow is to develop some intuition about uncovered sets. In doing so, we generalize to the spatial context results first reported by Miller (1980) for the case of tournaments over finite alternative sets.

A natural generalization of $UC(y)$ is

$$UC(X) = \{x \mid yCx \text{ for no } y \epsilon X\}.$$

$UC(X)$ is known in the literature (Miller, 1980; McKelvey, 1982a) as the *uncovered set*.

Trivially,

Lemma 6. $UC(X) = \bigcap_{y \epsilon X} UC(y)$.

Proposition 1 (Characterization of UC(y)).[16] $x \epsilon UC(y)$ if $x \epsilon W^2(y)$ and only if $x \epsilon R^2(y)$.

Proof. Necessity is given by Lemma 5. So, to establish sufficiency, assume $x \epsilon W^2(y)$. Then $\exists\ z \epsilon W(y)$ with $x \epsilon W(z)$. Thus $z \epsilon W(y) \cap W^{-1}(x)$ which, by Lemma 2b, implies $\sim yCx$. So $x \epsilon UC(y)$.

Q.E.D.

Without proof, we give an obvious extension and corollary.

Proposition 2 (Characterization of $UC(X)$). $x \epsilon UC(X)$ if $x \epsilon \bigcap_{y \epsilon X} W^2(y)$ and only if $x \epsilon \bigcap_{y \epsilon X} R^2(y)$.

Corollary. $UC(X) \cup UC(y)\ \forall\ y \epsilon X$.

Miller (1980) speculates that, when agents have Euclidean preferences, $UC(X)$ is a relatively small subset of the convex hull of agent ideal points. McKelvey's (1982a) results on bounds for $UC(X)$ in the Euclidean context

[16] The next two theorems and corollary are stated in a somewhat awkward form because of peculiarities on the boundaries of some of these sets.

substantiates this conjecture, though demonstrating that the size of $UC(X)$ depends on the degree of radial symmetry in agent ideal points (see note 11). Given the rather stringent intersection requirement of Proposition 2, our own intuition about $UC(X)$ is consistent with these other results.

By way of further elucidation we present one last characterization result.

Definition. $PO(X) = \{x \epsilon X \mid y\ P_i x\ \forall i \text{ for no } y \epsilon X\}$.

If agent preferences are Euclidean, then $PO(X)$, the Pareto optimal set, is the convex hull of agent ideal points.

Proposition 3. $UC(X) \subset PO(X)$.

Proof. Suppose $x \notin PO(X)$. Then there is some $y \epsilon X$ with $yP_ix\ \forall\ i \epsilon N$. Let $z \epsilon W(y)$ so that zP_iy for some majority. By A.2, P_i is transitive, so it follows that zP_ix for this same majority, namely, $z \epsilon W(x)$. Since z is an arbitrary element of $W(y)$, we have $W(y) \subset W(x)$. Together with $y \epsilon W(x)$, it follows that yCx. Thus $x \notin UC(y)$ and, by the Corollary to Proposition 2, $x \notin UC(X)$.

Q.E.D.

While the set of points not covered by any alternative is a subset of the Pareto optimals, the same is not true of $UC(y)$. As the examples that follow reveal, $UC(y) \cap PO(X)$ is nonempty and typically contains most of $PO(X)$. But "pieces" of $PO(X)$ may be covered by a given $y \epsilon X$, and many points not in $PO(X)$ are often not covered by y. However, we show in these examples that maximizing behavior by a centralized agenda setter leads back to $PO(X)$ so that it appears certain that the sophisticated outcome of any agenda selected by a centralized-agenda agent is Pareto optimal.[17]

In Figure 5 we have $N = (1, 2, 3)$ and $y \epsilon PO(X)$, where $PO(X)$ is the triangle connecting the three ideal points. We assume Euclidean preferences for the $i \epsilon N$. The three petal-shaped surfaces, labeled A, B, and C, comprise $W(y)$. The larger, more outlandishly shaped, surface containing A, B, and C is $UC(y)$. For any point x in this area, there is another point z, and an agenda $V = (y, x, z)$ with x the sophisticated agenda equilibrium. Thus by Theorem 3, the set of points that are saes of some agenda is quite large. Moreover, an agenda setter who must commence at y may nevertheless construct an agenda with any point in $UC(y)$ as the sophisticated result. Note in this example that $UC(y)$ is

[17] Ferejohn (personal communication, 1983) suggests that a related logic will imply Pareto optimal outcomes for the open-agenda process. Specifically, in analogy to the dynamic process modeled in Ferejohn, McKelvey, and Packel (1981), he conjectures that the opportunity for agents to add alternatives freely to the agenda will cause the set of points remaining uncovered to converge on $UC(X)$ which, by Proposition 3, is contained in $PO(X)$.

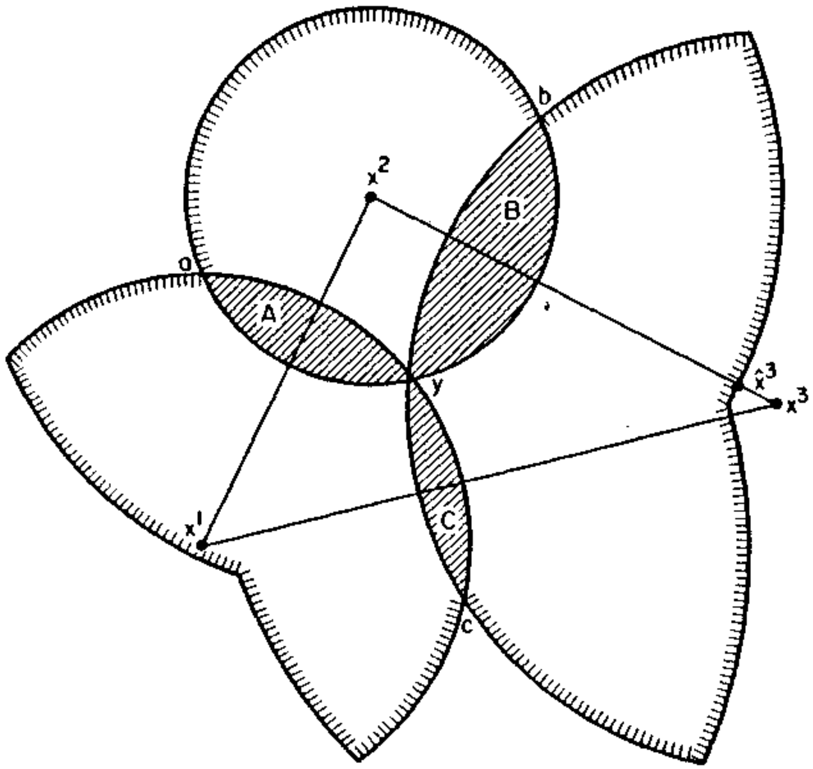

FIGURE 5

Uncovered Set for a Pareto Optimal Point

'large' in comparison to *PO*(*X*); it contains most of *PO*(*X*). Note, second, that it contains non-Paretian points. Finally, note that Mssrs. 1 and 2 can construct agendas, if either is the agenda setter, yielding their ideal points. Mssr. 3 can only obtain $\hat{x}^3$. In any of these cases, the sophisticated outcome is Pareto optimal (since Mssr. 3 will choose a point on the contract locus with one of the other agents).

Figure 6 analyzes a setting in which $y \notin PO(X)$. The two shaded petals constitute $W(y)$ and the oddly shaped outline containing $W(y)$ is $UC(y)$. Again note that some elements of $PO(X)$ are covered (though this time by a non-Pareto point) and that one prospective agenda setter will be unable to obtain his ideal point.

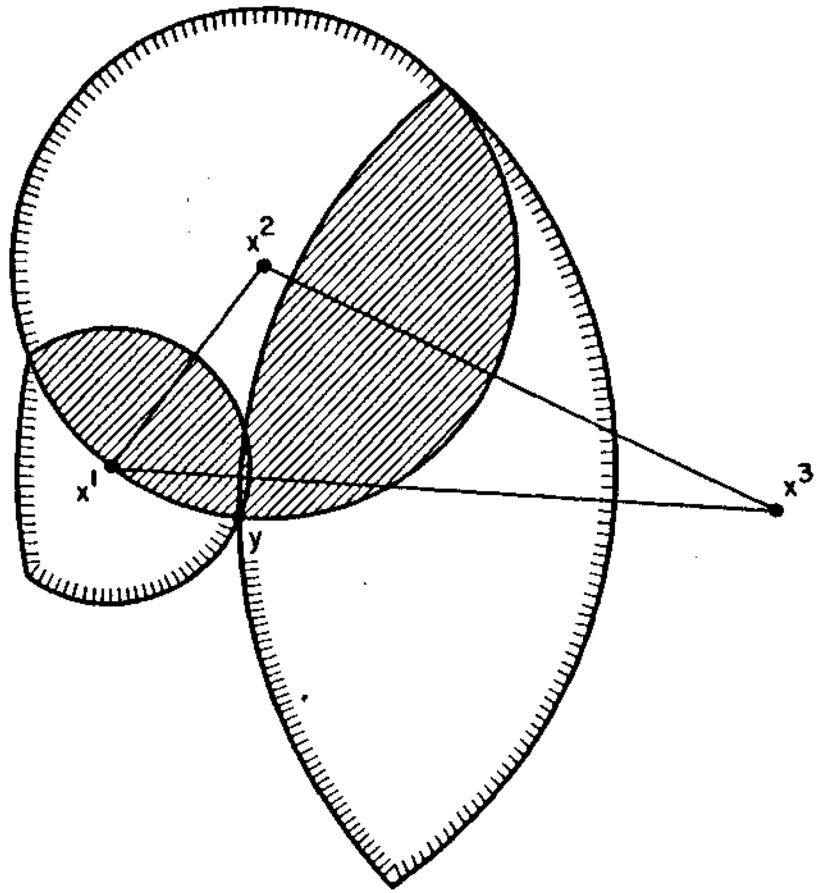

FIGURE 6

Uncovered Set for a Pareto-Dominated Point

These figures illustrate a result developed in McKelvey (1982a). The more extreme an agenda setter's ideal point is in comparison to the distribution of ideal points, the less likely it is to be part of $UC(X)$. On the other hand, in qualification, the more distant y is from $PO(X)$, the more likely it is that $PO(X) \subset UC(y)$.

5. Voting Outcomes under the Many Majority-Rule Institutions

In this paper, we have developed a new approach to sophisticated voting. The construction of the sophisticated agenda together with the intersecting-win-sets theorem provides a methodology for studying sequential-elimination majority-rule institutions. In this concluding discussion section, we apply this approach, in combination with the results in sections 3 and 4, to some issues in majority-rule voting. First we summarize our results.

Theorem 3 allows us to answer the question posed by McKelvey (1976) under conditions of sophisticated voting behavior: Where can majority rule wander, starting at some point y? The answer is that there exists an agenda to any point (and only those points) not covered by y. This implies that no matter what the agenda-formation process, the sophisticated outcome cannot lie outside of $UC(y)$. Clearly, this also completely answers McKelvey's question for a centralized agenda setter, with monopoly power over construction of V, operating in an environment of sophisticated agents.[18]

This discussion suggests two dimensions characterizing the set of agenda institutions: (1) centralized versus noncentralized agenda agents and (2) sincere versus sophisticated voting agents. The set of agenda equilibria given centralized agenda setting and sincere voters is X itself (McKelvey's Theorem). From any commencement point $y \epsilon X$, there exists an agenda that yields any $x \epsilon X$ as the sincere agenda equilibrium. Alternatively the set of agenda equilibria, given centralized agenda setting and sophisticated voters, is $UC(y)$ for any commencement point $y \epsilon X$ (Theorem 3).

What of majority rule without centralized agenda setting? In particular, consider an "open" or completely decentralized agenda mechanism according to which any agent may add an alternative to the voting order. No equilibrium exists with sincere agents. If the agenda $V = (x_1, \ldots, x_i, \ldots, x_n)$ has x_i as the sincere majority rule outcome, then, since $W(x_i) \neq \phi$, there is always an agent with incentive to move $x_{n+1} \in W(x_i)$ thus producing x_{n+1} as the new sincere outcome. But the new agenda, $V' = (x_1, \ldots, x_i, \ldots, x_n, x_{n+1})$ is not an "equilibrium agenda" for the same reason (Ferejohn, Fiorina, McKelvey, 1981). And so on. Thus completely decentralized agenda formation with sincere voters is ill behaved. Final outcomes will depend upon an arbitrary stopping rule. The set of potential outcomes is X itself.

[18] Theorem 3 characterizes the feasible set over which the centralized agenda agent optimizes. If his ideal point is not covered by y, then he will construct an agenda yielding it as the voting result. If his ideal point is covered by y, then he is constrained to choose the point that he most prefers in the feasible set. Whatever his optimum in this set, he can obtain it from an agenda in which y is the first element and there are no more than two divisions (Corollary 3.1).

Our results above have something to say about open-agenda processes with sophisticated voters. Theorem 3 states that if x and y are placed on the agenda, and y covers x, then x can never be the sophisticated outcome. Even if $\sim yCx$, so long as there exists some $z \epsilon X$ with zCx, that is, so long as $x \notin UC(X)$, some agent may veto x by placing z on the agenda. Thus it appears that the relevant set of possible outcomes for open-agenda processes with sophisticated voters is $UC(X)$.[19] From Proposition 3, this institutional arrangement assures a Pareto optimal outcome.

The literature on voting models treats majority-rule institutions holistically and indiscriminately. Yet, an important point emerging from our discussion is that there are many majority-rule institutions. While all are based on the majority preference relation, this is surely not the only influence over outcomes. The degrees of agenda centralization and agent voting sophistication are two important additional features of such institutions. Even these, however, do not exhaust the possibilities. There is a third dimension—building agendas "forward" versus "backward" (see footnote 3). Thus the set of majority-rule institutions covers a large variety of voting games with different characteristics. Since each is based on majority rule, the general results about the majority-rule preference relation apply. But only in the simplest of institutions (no agenda controls, sincere voting, agenda built forward) does the cyclicity of the majority-rule preference relation directly characterize outcomes. What may be concluded from this discussion is that researchers, having exhaustively characterized the majority-rule preference relation, should turn their attention to unexplored aspects of the set of institutions based on majority rule. To illustrate this point, we conclude the paper by studying the different restrictions on outcomes implied by building the agenda forward rather than building it backward. This development displays the real power of the intersecting-win-sets theorem as a tool for characterizing the set of sophisticated outcomes. As we have seen, this theorem determines the sae for any given agenda. But it tells us more. It also specifies what restrictions on the location of the final outcome are implied by the presence of any set of alternatives on the agenda.

All the presentations in the literature cited in this paper (with the exception of Ferejohn, Fiorina, and McKelvey, 1981) assume that a majority-rule process begins at some point $y \epsilon X$, and seek to determine where the process can go from there. The agenda, $V = (x_1, \ldots, x_n)$ is assumed to be built forward, with x_1 moved first, x_2 next, and so on, with voting then proceeding in precisely the same order. Building the agenda in reverse, though equally plausible, has received scant attention (Shepsle and Weingast, 1981, 1982). Yet, most real-world legislatures and

[19] McKelvey (1982a) comes to a similar conclusion, and this appears to be what Miller (1980) had in mind.

committees embody agenda mechanisms of this latter type. Most, for example, provide that the status quo, x^0, be voted last. Others impose additional restrictions like designating some privileged alternative (say a committee bill) to be in the penultimate division. While any number of amendments may subsequently be moved, they are voted on *before* x^0, rather than after x^0. Thus $V = (x_1, \ldots, x_n)$ is constructed by first naming x_n, then x_{n-1}, and so on.

To illustrate this point, let us return to the example with which we began the paper. This time, however, let us set down the rules of agenda procedure (typical of legislatures like the U.S. House of Representatives) that result in building an agenda backward:

1. There is a status quo ante (labeled x_1), which is voted on last in the sense that some other alternative (possibly perfected by amendments), in order to be the final choice from V, must avoid losing to x_1 at the last vote.[20]

2. A bill or motion (labeled x_2) is the legislative vehicle for altering the status quo, x_1. It enters the voting at the penultimate division and, if it passes, is paired against x_1 in the final division.

3. At most four amendments are in order (according to the rules of the U.S. House): an amendment to the bill (x_3), an amendment to this amendment (x_4), a substitute (x_5), and an amended substitute (x_6).

4. Thus at any one time there may be as many as six motions pending, $V = (x_6, x_5, x_4, x_3, x_2, x_1)$.

Notice, then, that the agenda formed is like that in the example of Figures 2 and 3, but the votes are taken in the reverse order since the agenda is now built backwards.

The sincere outcome of this agenda is given directly by the majority preference relations, P. For the one illustrated in Figure 3, we have x_6 defeating x_5, x_6 losing to x_4, x_4 losing to x_3, and x_3 beating both x_2 and x_1. So x_3 is the sincere agenda equilibrium of the built-backward agenda V.[21] The sophisticated outcome is obtained from Theorems 1 and 2 by constructing the sophisticated agenda, Z. From the algorithm given earlier, $Z = (x_4, x_1)$. So, by Theorem 1, x_4 is the sae.[22]

We may study this issue more generally: What is the difference between building forward and building backward? The intersecting-win-sets theorem implies that with sophisticated voters, an agenda built backward is much more constraining than the same agenda built forward. That is, if we commence with y, then the sophisticated outcome is an

[20] Parliament men usually describe this as a "vote on final passage," a "motion to table," a "motion to recommit," or a "motion to strike the enacting clause." Any of these motions has the effect of pitting a perfected bill against the status quo ante, x_1.

[21] Note that the same agenda, built forward, yielded x_4 as the outcome.

[22] Built forward, this agenda earlier yielded x_3 as the sae.

element of $UC(y)$. If, on the other hand, y is the final element of an agenda, then, no matter what additional elements precede it, the sophisticated outcome is either y or an element of $W(y)$.

To see this let V_y and V'_y be two families of agendas. The former consists of all agendas built forward from y while the latter consists of the same agendas built backward from y. Let $SAE(V_y)$ and $SAE(V'_y)$ be the sets of sophisticated agenda equilibria of the two families. Thus $SAE(V_y)$ consists of those alternatives that are the sae of some agenda that commences at y, while $SAE(V'_y)$ consists of the saes of agendas terminating at y.

Theorem 4 (Forward vs. Backward Agendas). $SAE(V'_y) \subset SAE(V_y)$.
Proof: From Theorem 2, $SAE(V'_y) \subset W(y)$. From Lemma 4, $W(y) \subset W^2(y)$ and, from Proposition 1, $W^2(y) \subset UC(y)$. But, from Theorem 3, $SAE(V_y) = UC(y)$. Thus, $SAE(V'_y) \subset SAE(V_y)$.
Q.E.D.

Building forward from y, the sophisticated outcome is contained in $UC(y)$. Building backward it is contained in $W(y)$. And the latter is contained in the former (see Figures 5 and 6 for illustrations).

Theorem 4 generalizes in an obvious way. If we fix m elements of an agenda, $y_1, \ldots, y_m$, then the feasible sophisticated outcomes based on building backward from these m elements are contained in the set of feasible sophisticated outcomes based on building forward from those fixed elements. In sum, building backwards from fixed elements constrains outcomes more than building forward.[23]

The main point of our results is that institutions impose constraints on agenda formation and that these have systematic implications for outcomes under majority rule. Different sets of restrictions, because they imply different sets of feasible agendas, imply different sets of potential outcomes. Thus in our view the most fruitful way to proceed in the theory of majority voting (with an eye toward understanding legislative and committee institutions) is to study institutional restrictions on agenda formation and to show their resulting effects on outcomes. The tools developed in this paper, in particular the intersecting-win-sets theorem, should play a central role in this endeavor.

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[23] Analogous results may be obtained for sincere agents so that McKelvey's (1976) development is subject to the same caveat, viz., building the agenda backwards—even with sincere voters—is more constraining on a centralized agenda setter than building it forwards.

Appendix

Proof of Theorem 1 (Equivalence Theorem)

In this Appendix we prove Theorem 1 (equivalence theorem). We employ the following notational conventions and definitions:

> $se(V)$ = sophisticated equivalent of agenda V.
> sae (V) = sophisticated agenda equilibrium, the first element of the sophisticated agenda, of the agenda V.
> $Y(j)$ = agenda of length of j, viz. $(y_j, y_{j-1}, \ldots, y_1)$.
> $D(j;j-1)$ = agenda $Y(j)$ with y_{j-1} deleted, namely, $(y_j, y_{j-2}, \ldots, y_1)$.

Equivalence Theorem. For the amendment process, the sophisticated equivalent of an agenda V is identical to the first element of the sophisticated agenda, that is, $se(V)$ = sae (V).

Proof. We proceed by induction on the length of agenda V.

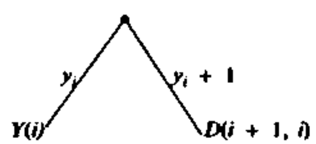
Step 1. Trivial for $Y(1)$.

Step 2. Assume true for all agendas of length i: Let $y_k \in Y(i)$ be the se of $Y(i)$. Let the associated sophisticated agenda be $Z = (z_i \ldots z_1)$. Note that it is the same length as $Y(i)$ in accord with the algorithm of the text. The induction hypothesis, se$[Y(i)]$ = sae$[Y(i)]$, requires that $z_1 = y_k$.

Step 3. Having assumed the hypothesis for agenda of length i, we now show that it is true for agendas of length $i + 1$.

Without loss of generality, we show the result for the agenda $Y(i + 1)$. Our strategy of proof is to decompose this agenda into its components so we can calculate its se and sae. Using Step 2, we show they are always the same.

The tree associated with $Y_{(i+1)}$ is



Notice that $Y(i)$ and $D(i + 1, i)$ are identical except for the first element:

$$Y(i) = [y_i, Y(i - 1)]$$
$$D(i + 1, i) = [y_{i+1}, Y(i - 1)]$$

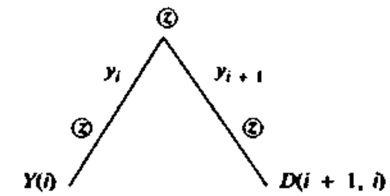
Let z be the sae of $Y(i - 1)$. By the induction hypothesis for agenda of length i, se$[Y(i)]$ = sae$[Y(i)]$. It is either y_i or z (and is y_i iff $y_i \in W(z_j)$, $\forall z_j$ on the sophisticated agenda of $Y(i)$). Similarly, se$(D(i + 1, i))$, = sae$(D(i + 1, i))$, since it is an agenda of length i. It is either y_{i+1} or z (and is y_{i+1} iff $y_{i+1} \in W(z_j)$ $\forall z_j$ on the sophisticated agenda of $D[i + 1, i]$). Notice that under the amendment process: (1) any agenda of length $i + 1$ pits the winner of two agendas of length i; and (2) these agendas differ by only one element.

Thus the agenda $Y(i + 1)$ pits the winner of $Y(i)$ against the winner of $D(i + 1, i)$. Since the winner of each of the latter two agendas can be only one of two elements (z *or* y_i, and z or y_{i+1}, respectively), we can calculate both the se$[Y(i + 1)]$ and sae$[Y(i + 1)]$ in all four cases. We show that the induction hypothesis implies that these are identical.

The four cases to consider depend upon what happens in $Y(i)$ and $D(i + 1, i)$.

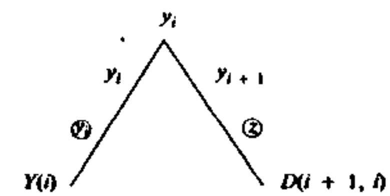
Case	se$(Y(i))$ = sae $(Y(i))$	se$(D(i + 1, i))$ = sae$(D(i + 1, i))$
1	z	z
2	y_i	z
3	z	y_{i+1}
4	y_i	y_{i+1}

Case 1.



We are given that $y_i \neq$ se$[Y(i)]$ and $y_{i+1} \neq$ se$[D(i + 1, i)]$. The former implies that $y_i \notin W(z_j)$ for some z_j on the sophisticated agenda of $Y(i)$, and the latter that $y_{i+1} \notin W(z_j)$ for some z_j on the sophisticated agenda of $D(i + 1, i)$. Thus z is the first element of the sophisticated agenda of $Y(i + 1)$ since neither y_i nor y_{i+1} beat all other elements on the sophisticated agenda of $Y(i + 1)$. But z = se$[Y(i + 1)]$ since, being the *se* at both lower nodes, it advances to the higher node. Therefore, sae$[Y(i + 1)]$ = se$[Y(i + 1)]$.

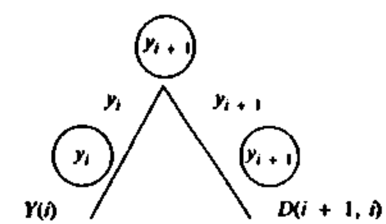
Case 2.



y_i = se$[Y(i)]$, z = se$[Y(i - 1)]$, and z = se$[D(i + 1, i)]$ are given. The former implies $y_i \in W(z)$ so that y_i = se$[Y(i + 1)]$. We must show y_i = sae$[Y(i + 1)]$. Since y_i = se$[Y(i)]$, by the induction hypothesis, y_i = sae$[Y(i)]$. This implies it is also on the sophisticated agenda of $Y(i + 1)$ {since $Y(i + 1) = [y_{i+1}, Y(i)]$}. Now, y_{i+1} can be sae$[Y(i + 1)]$ only if it beats all other elements on the sophisticated agenda. But $y_{i+1} \neq$ sae$[D(i + 1, i)]$ $\rightarrow \exists$ some element on the sophisticated agenda of $Y(i - 1)$ that it does not beat (since $D(i + 1, i) = [y_{i+1}, Y(i - 1)]$). Therefore y_{i+1} is not on the sophisticated agenda of $Y(i + 1)$ and y_i is first element, that is, y_i = sae$[Y(i + 1)]$.

Case 3. Same as Case 2.

Case 4.



Without loss of generality, assume

$$y_{i+1} \text{ P } y_i.^*$$

Then y_{i+1} = se[$Y(i+1)$] since, by definition, the se is the sincere winner between two lower se's and, by (*), y_{i+1} beats y_i. By induction hypothesis, y_i = sae[$Y(i)$] and y_{i+1} = sae [$D(i+1, i)$]. This implies y_{i+1} beats all elements of the sophisticated agenda of $Y(i-1)$. By (*) it also beats y_i. Hence $y_{i+1} \in W(y_i) \cap W(z_j)\ \forall\ z_j$ on sophisticated agenda of $D(i+1, i)$. This implies y_{i+1} is first element of the sophisticated agenda of $Y(i+1)$, that is, y_{i+1} = sae$Y$[$i+1$)]. $\therefore$ sae[$Y(i+1)$] = se[$Y(i+1)$].

Q.E.D.

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*Policy Reasoning and Political Values: The Problem of Racial Equality**

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What is the structure of policy reasoning among citizens at large, and particularly, how does this structure vary with the level of education? To answer this question, we examine the nature of policy reasoning on the issue of racial equality. Our analysis helps explain why the highly educated show greater support for the principle of racial equality than do the less educated but not appreciably greater support for government efforts to promote it. Highly educated citizens, we argue, have more fully integrated and differentiated belief systems, and thus they take a wider range of factors into account when evaluating government policy.

Americans seemingly have a weak grip on democratic values. The root difficulty is not that people reject such values—on the contrary, nearly all accept them, stated in the abstract—but that they are not ready to stand by basic principles in specific controversies. This gap between abstract and concrete reflects, it is commonly agreed, a certain lack of thoughtfulness about politics on the part of mass publics, a failure to reason from the general to the specific. We shall suggest, however, that the cause of this gap is not always thoughtlessness; that indeed it may be just the opposite: what *appears* as a gap for some Americans results precisely from their being thoughtful.

To develop this thesis, we examine the nature of policy reasoning among citizens at large. Specifically, the study reported herein focuses on the process by which people translate commitment to racial equality at the level of principle into support for it at the level of policy. Our analysis of this process helps explain why education may build support for the value of equality without at the same time increasing support for policies designed to realize it.

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