The Durable Monopoly Game, Part I: Theory and the Anti-Coase Conjecture

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Abstract. The durable monopoly game with uniformly distributed consumers is studied within a finite time horizon. It is shown that the game generically possesces a unique subgame perfect equilibrium which can be explicitely calculated. To test the Coase-Conjecture the finite time horizon is divided into successively more subperiods and the limiting solution is derived. It turns out that the Coase-Conjecture does not hold.

1. Introduction

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2. The Model

The game to be studied in the sequel is a T-stage game, $1 \leq T < \infty$. To begin with the number of stages or periods, T, is taken finite and is thought of as the number of subperiods, say, "market days", within a finite interval of time. Eventually, however, this number of market days within a finite time interval will be increased and the limit of the solutions, when T approaches infinity, will be studied.

The game is one between a single monopolist supplier of a durable and indivisible commodity, and a large number of different consumers. All the potential consumers have a "willingness to pay" or reservation value v for one unit of the durable good, i.e. a customer wants either one unit of the commodity or none. Without loss of generality it can be assumed that 1 is the highest and 0 the lowest reservation value, such that reservation values v for all consumers satisfy $0 \leq v \leq 1$. It is assumed that for every number $v \in [0, 1]$ there exists exactly one potential customer with reservation value v and that this uniform distribution of reservation values over the unit interval is common knowledge.

For the monopolist the marginal cost of production is assumed constant. Let p_t denote the unit profit in period t, $t = 1, \ldots, T$, that is: period t's sales price minus the constant marginal cost of production. In the sequel p_t will simply be called the price in period t, because marginal to be

costs can be assumed zero without loss of generality. If x_t denotes the number of units sold in period t, the monopolist's profit in period t, π_t, is given by $\pi_t = p_t x_t$. The monopolist's time preference is given by a constant discount factor $\rho \in (0, 1)$, such that the monopolist's payoff function for the game can be described by

In a subgame starting in a period $t > 1$ the profits $p_\tau x_\tau$ of periods $\tau < t$ are, of course, given so that the monopolist will try to maximize

$$\Pi_1 = \sum_{t=1}^{T} \rho^{t-1} p_t x_t .$$

$$\Pi_t = \sum_{\tau=t}^{T} \rho^{\tau-1} p_\tau x_\tau .$$

Since a consumer with reservation value v will buy at most one unit, his payoff in the case, where he never buys, can be normalized to zero. If he does buy one unit in period t at the price p_t, the payoff to the consumer with reservation value v (referred to as consumer v) is given by $\delta^{t-1}(v - p_t)$, where $\delta \in (0, 1)$ is the common discount factor of consumers. The reservation value v can be thought of as the discounted stream of benefits to consumer v from enjoying the durable good from period t onwards. At the price $p_t = v$ consumer v is indifferent between buying and not buying. The payoff function u_v of the potential customer v is thus given by

$$u_v = \begin{cases} \delta^{t-1}(v - p_t), & \text{if } v \text{ buys in } t, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that a consumer can buy only once and after having bought, he can neither resell, nor participate actively in the game in any other way.

To complete the definition of the extensive form game it remains to specify the market process together with the information requirements. In every period t, $t = 1, \ldots, T$, the monopolist moves first by setting a price $p_t \in [0, 1]$. Then, observing the price, all potential customers, who are still active (have not bought yet in any previous period) choose simultaneously whether to buy or to continue waiting. At the end of the period the outcome of the market in period t becomes publicly known. Since perfect recall is assumed, any player in any period knows all ~~the history~~ previous moves leading to the current subgame.

This completes the description of the durable monopoly game. The situation envisaged by Coase [1972] can be represented by setting $\delta = \exp(-r\Delta)$, $\rho = \exp(-R\Delta)$, $r > 0$, $R > 0$, where $\Delta = 1/T$ is the length of market days, and considering the limit as T approaches infinity.

3. Results

The focus of this section is to derive ~~the~~ two major results on the durable monopoly game with a finite time horizon. First it will be shown that (under a certain refinement of subgame perfection) there exists a unique equilibrium outcome of the durable monopoly game. Second the limiting behavior of the equilibrium outcome ~~as~~ $1/T \to 0$ will be studied.

The technique of the proofs rests on two cornerstones. The first, Lemma 1, is a characterization of the set of all subgame perfect equilibria of the game: It says that, under very weak conditions, for each period t the set of still active consumers along the equilibrium path is an interval with lower bound zero and an upper bound which is decreasing over time. The second cornerstone is a sequence of recursively defined coefficients, the properties of which will be described in Lemmas 2 and 3. Since the proofs of these two Lemmas require lengthy induction arguments, the proofs are relegated to the Appendix.

Concerning the first lemma, there will be two notions used which require clarification. Recall that the strongest notion of Nash-refinements is a *strict* equilibrium, i.e. a Nash equilibrium, where the equilibrium strategy of each player is the unique best response to the equilibrium strategies of the other players. In games with a continuum of players this is perhaps too much to ask - ~~for example~~ in the durable monopoly game there will ~~for each choice of p~~ always be one consumer, who is indifferent between buying ~~and not buying, namely $v = p$~~. For this class of games it is therefore natural to weaken the notion of a strict equilibrium slightly and require that equilibrium strategies are unique best responses for all players except, possibly, for a null set. ~~Thus defined~~ Nash equilibrium ~~to be~~ *strict with respect to almost all players*, if the equilibrium strategy is the unique best response to the equilibrium strategies of the other players for all players except for a closed set of consumers with (Lebesgue-) measure zero. Observe that this definition implicitely assigns positive mass to the monopolist.

situation at hand we say that

For instance,

The second notion in Lemma 1 which requires specification is that the Lemma states that its conclusion applies "generically" to all subgame perfect equilibria of the durable monopoly game. Such a notion is usually best defined by stating, when a property is "non-generic". We will say that some property of the game is *non-generic*, if it is possible to find a continuous (cumulative) distribution function on the unit interval ~~for the consumers' distribution~~ which is arbitrary close to the uniform distribution (say, in the supremum-norm), but which destroys the property (if it is substituted for the uniform distribution).

I am not too happy about this! (why don't we restrict our deviations to nearly all customers, i.e. up to null sets?

the set of possible reading values,

Most of the notation will be developed, where it is needed. For the time being, let, for any given subgame perfect equilibrium σ, the set of consumers $v \in [0, 1]$, who are still active in period t along the equilibrium path induced by σ (because σ prescribes that they do not buy before period t, unless a deviation occurs), be denoted by $\mathcal{V}_t(\sigma) \subset [0, 1]$.

of players whose equilibrium strategy is not the only best response on the equilibrium play

LEMMA 1. *For any subgame perfect equilibrium σ which is strict with respect to almost all players and generically for all subgame perfect equilibria σ of the durable monopoly game,*

| with T ?

$$\text{int} \mathcal{V}_t(\sigma) = (0, v_t(\sigma)) \quad v_1(\sigma) = 1$$

and $0 \le v_{t+1}(\sigma) \le v_t(\sigma) \le 1$, *for all* $t = 1, \ldots, T-1$.

PROOF: For some given subgame perfect equilibrium σ let $p(\sigma) \in \Re^T_{++}$, $p(\sigma) = (p_1(\sigma), \ldots, p_T(\sigma))$ denote the sequence of prices induced along the equilibrium path. Denote by (σ_{-v}, w_t) the strategy combination induced by σ in the subgame after in period t a consumer v, who was supposed to buy in t under σ, has deviated to waiting, $v \in \mathcal{V}_t(\sigma)$.

Consider some $v \in \mathcal{V}_t(\sigma)$, who in equilibrium does not buy in period t (at the price $p_t(\sigma)$). For this $v \in \mathcal{V}_t(\sigma)$ it must either be true that $v \le p_{t+k}(\sigma), \forall k = 0, \ldots, T-t$, or there must exist some $k = 1, \ldots, T-t$, such that

$$v - p_t(\sigma) \le \delta^k(v - p_{t+k}(\sigma))$$

$$\iff \quad v \le \frac{p_t(\sigma) - \delta^k p_{t+k}(\sigma)}{1 - \delta^k}.$$

Consequently, for all $v' < v$, $v' \in \mathcal{V}_t(\sigma)$, either $v' < p_{t+k}(\sigma)$, $\forall k = 0, \ldots, T-t$, or

$$v' < \frac{p_t(\sigma) - \delta^k p_{t+k}(\sigma)}{1 - \delta^k}$$

$$\iff v' - p_t(\sigma) < \delta^k(v' - p_{t+k}(\sigma)),$$

for at least one $k = 1, \ldots, T-t$. We wish to show that all $v' < v$ will not buy either in equilibrium. Suppose some $v' < v$ does buy in equilibrium. Then $v' < p_t(\sigma)$ is impossible, and it must be true that there exists some $k = 1, \ldots, T-t$ such that

in period t.

H period t according to σ.

$$0 \le v' - p_t(\sigma) < \delta^k(v' - p_{t+k}(\sigma)),$$

$$\text{and} \quad v' - p_t(\sigma) \ge \delta^l(v' - p_{t+l}((\sigma_{-v'}, w_t))),$$

for all $l = 1, \ldots, T-t$. But these two inequalities imply

$$p_{t+k}((\sigma_{-v'}, w_t)) > p_{t+k}(\sigma)$$

for at least one $k = 1, \ldots, T-t$. Since $(\sigma_{-v'}, w_t)$ differs from σ only by an individual deviation (of mass zero), the sequence of ~~masses~~ residual demands, i.e. the set of all still active consumers)

$(\int_{\mathcal{V}_\tau(\sigma)} dv)_{\tau=t}^{T}$, on which the monopolist's pricing decisions ~~must~~ are by subgame perfection ~~be~~ based, remains unchanged by the deviation. Hence, that for some k *two* prices can be subgame perfect equilibrium choices (in the subgame), implies that in period $t+k$ the monopolist has two distinct best responses to the consumers' equilibrium strategies. This certainly contradicts the requirement that the equilibrium be strict with respect to almost all players. Thus consider subgame perfection without the strictness requirement: Since the monopolist's profit function of period $t+k$ always contains a term of the form

why can't that trigger further deviations due to appropriate [?] strategies

$$\int_{\mathcal{V}_{t+k}(\sigma)\cap(p_{t+k},\infty)} dv\,,$$

the property that the monopolist has two distinct best responses can always be destroyed by choosing a slightly different distribution function of consumers close to the uniform (twist in on $(0,1)\setminus\mathcal{V}_{t+k}(\sigma)$, where the one-period profit is linear in p_{t+k}).[1]

We conclude that, if $v \in \mathcal{V}_t(\sigma)$ does not buy in period t, then any $v' < v$ will not buy either, under the two alternative hypotheses of the Lemma. It follows that $\mathcal{V}_{t+1}(\sigma)$ is an interval, if $\mathcal{V}_t(\sigma)$ is an interval, such that

$$\operatorname{int}\mathcal{V}_{t+1}(\sigma) = (\inf\mathcal{V}_t(\sigma),\ \max_{k=1,\ldots,T-t}\frac{p_t(\sigma)-\delta^k p_{t+k}(\sigma)}{1-\delta^k}),$$

for all $t = 1,\ldots,T-1$. Now clearly, $\mathcal{V}_1(\sigma) = [0,1]$, $v_1(\sigma) = 1$, $\inf\mathcal{V}_{t+1}(\sigma) = \inf\mathcal{V}_t(\sigma)$, imply $\inf\mathcal{V}_t(\sigma) = 0$, $\forall t = 1,\ldots,T$, and $v_{t+1}(\sigma) = \sup\mathcal{V}_{t+1}(\sigma) \le \sup\mathcal{V}_t(\sigma)$ implies $0 \le v_{t+1}(\sigma) \le v_t(\sigma) \le 1$, as required. Of course, this was derived under the assumption that not all $v \in \mathcal{V}_t(\sigma)$ buy in period t in equilibrium. If they do, then $\mathcal{V}_{t+1}(\sigma) = \emptyset$, $v_{t+1}(\sigma) = 0$, verifies the claim of the Lemma. ∎

Subgame perfection enters the proof of Lemma 1 in an essential way, because it is required to keep the monopolist from threatening with "punishment" price choices which are not or not generically best responses at the corresponding information sets. A short comment on the notion of a generic game, as used above, may be appropriate here also: The reader may wonder, why "non-generic" was defined in terms of the distribution rather than, say, in terms of discount factors. The reason for this is simply that discount factors have no rolë to play in the final period $t = T$. And although one could presumably apply similar arguments in terms of discount factors for any period $t < T$, this does not

[1] Strictly speaking this argument should be iterated backwards, starting from period $t = T-1$. For the sake of brevity, we have avoided that.

from the above, which implies

$$
\begin{aligned}
1-\delta+\left(\frac{\delta}{2}-\frac{\rho}{2}\right)a_t &> \frac{1}{2}(1-\rho)a_t > 0 \implies \\
\implies 1-\delta+\frac{\delta}{2}a_t &> \frac{\rho}{2}a_t > \frac{\rho}{4}a_t\,,
\end{aligned}
$$

as required. ∎

PROOF OF LEMMA 3: (i) From $p_t = F_t(p_{t-1})$ one has

$$
\begin{aligned}
\frac{p_{t-1}-\delta p_t}{1-\delta} &= (1-\delta)^{-1}\left[1-\frac{\frac{\delta}{2}a_t}{1-\delta+\frac{\delta}{2}a_t}\right]p_{t-1} = \\
&= \frac{p_{t-1}}{1-\delta+\frac{\delta}{2}a_t}\,.
\end{aligned}
$$

Consequently, the inequality (i) holds, if and only if

$$
\begin{aligned}
&\frac{p_{t-2}}{1-\delta+\frac{\delta}{2}a_{t-1}} > \frac{p_{t-1}}{1-\delta+\frac{\delta}{2}a_t} = \frac{\frac{1}{2}a_{t-1}p_{t-2}}{(1-\delta+\frac{\delta}{2}a_t)(1-\delta+\frac{\delta}{2}a_{t-1})} \\
&\iff 1-\delta+\frac{\delta}{2}a_t > \frac{1}{2}a_{t-1}\,.
\end{aligned}
$$

Using the definition of a_{t-1} the latter inequality reads

$$
\begin{aligned}
1-\delta+\frac{\delta}{2}a_t &> \frac{1}{2}\,\frac{(1-\delta+\frac{\delta}{2}a_t)^2}{1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_t} \iff \\
1-\delta+\left(\frac{\delta}{2}-\frac{\rho}{4}\right)a_t &> \frac{1}{2}\left(1-\delta+\frac{\delta}{2}a_t\right) \iff \\
1-\delta+\frac{\delta}{2}a_t &> \frac{\rho}{2}a_t\,,
\end{aligned}
$$

which is Lemma 2, (ii).

(ii) The definition of F_t implies that

$$
p_{t+k} = \prod_{j=0}^{k-1} \frac{\frac{1}{2}a_{t+k-j}}{1-\delta+\frac{\delta}{2}a_{t+k-j}}\,p_t\,, \quad \forall k = 1,\ldots,T-t,
$$

such that

$$
\frac{p_t-\delta^k p_{t+k}}{1-\delta^k} > \frac{p_t-\delta^{k+1}p_{t+k+1}}{1-\delta^{k+1}}
$$

can equivalently be written as

$$\Big(\frac{1-\delta^k 2^{-k}\prod_{i=0}^{k-1}(1-\delta+\frac{\delta}{2}a_{t+k-i})^{-1}a_{t+k-i}}{1-\delta^k}\Big)p_t >$$
$$> \Big(\frac{1-\delta^{k+1}2^{-k-1}\prod_{i=0}^{k}(1-\delta+\frac{\delta}{2}a_{t+k+1-i})^{-1}a_{t+k+1-i}}{1-\delta^{k+1}}\Big)p_t$$

which is equivalent to

$$1-\delta^{k+1}-\delta^k(1-\delta^{k+1})2^{-k}\prod_{i=0}^{k-1}(1-\delta+\frac{\delta}{2}a_{t+k-i})^{-1}a_{t+k-i} >$$
$$> 1-\delta^k-\delta^{k+1}(1-\delta^k)2^{-k-1}(1-\delta+\frac{\delta}{2}a_{t+k+1})^{-1}a_{t+k+1}\times$$
$$\times\prod_{i=0}^{k-1}(1-\delta+\frac{\delta}{2}a_{t+k-i})^{-1}a_{t+k-i}$$

which, finally, is equivalent to

$$2^k\prod_{i=0}^{k-1}\frac{1-\delta+\frac{\delta}{2}a_{t+k-i}}{a_{t+k-i}} > \frac{1-\delta^{k+1}}{1-\delta} -$$
$$-\delta\frac{1-\delta^k}{1-\delta}\,\frac{\frac{1}{2}a_{t+k+1}}{1-\delta+\frac{\delta}{2}a_{t+k+1}} = \frac{1-\delta^{k+1}+\frac{\delta^{k+1}}{2}a_{t+k+1}}{1-\delta+\frac{\delta}{2}a_{t+k+1}}.$$

The latter inequality will now be demonstrated by induction. Let $k=1$; then

$$2\frac{1-\delta+\frac{\delta}{2}a_{t+1}}{a_{t+1}} > \frac{1-\delta^2+\frac{\delta^2}{2}a_{t+2}}{1-\delta+\frac{\delta}{2}a_{t+2}} \iff$$
$$\frac{2(1-\delta)[1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_{t+2}]+\delta[1-\delta+\frac{\delta}{2}a_{t+2}]^2}{1-\delta+\frac{\delta}{2}a_{t+2}} > 1-\delta^2+\frac{\delta^2}{2}a_{t+2}$$
$$\iff 1-\delta+\frac{\delta}{2}a_{t+2} > \frac{\rho}{2}a_{t+2},$$

which is Lemma 2, (ii). Next assume

$$2^{k-1}\prod_{i=0}^{k-2}\frac{1-\delta+\frac{\delta}{2}a_{t+k-1-i}}{a_{t+k-1-i}} > \frac{1-\delta^k+\frac{\delta^k}{2}a_{t+k}}{1-\delta+\frac{\delta}{2}a_{t+k}}.$$

rule out the possibility that the monopolist has two best responses in the final period. (Strictness with respect to almost all players, of course, rules this out immediately.)

The next ingredient to the analysis is an, at this stage, purely formal definition of a sequence of coefficients. Fix some T, $1 \leq T < \infty$, and define recursively the sequence $(a_t)_{t=1}^{T}$ of coefficients by

$$a_t = \frac{(1-\delta+\frac{\delta}{2}a_{t+1})^2}{1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_{t+1}}; \quad a_T = 1,$$

for $t = 1, \ldots, T-1$ and for $\delta \in (0,1)$ and $\rho \in (0,1)$. The properties of these coefficients are given in the next two lemmas, the proofs of which can be found in the Appendix.

LEMMA 2.

is (i) needed ?!

(i) $1-\delta+\frac{\delta}{2}a_t > \frac{\rho}{4}a_t, \forall t = 1, \ldots, T;$

(ii) $1-\delta+\frac{\delta}{2}a_t > \frac{\rho}{2}a_t, \forall t = 1, \ldots, T;$

(iii) $0 < a_t < 2, \forall t = 1, \ldots, T.$

(PROOF: see Appendix.)

Using these coefficients now define the sequence of functions $(F_t)_{t=1}^{T}$, $F_t \colon \Re_{++} \to \Re_{++}$, by

$$F_t(p) = \frac{\frac{1}{2}a_t}{1-\delta+\frac{\delta}{2}a_t}p.$$

LEMMA 3. *If* $p = (p_1, \ldots, p_T) \in \Re_{++}^{T}$ *satisfies* $p_t = F_t(p_{t-1})$, $\forall t = 2, \ldots, T$, *then*

(i) $\dfrac{p_{t-1}-\delta p_t}{1-\delta} < \dfrac{p_{t-2}-\delta p_{t-1}}{1-\delta}, \forall t = 3, \ldots, T;$

(ii) $\dfrac{p_t-\delta^k p_{t+k}}{1-\delta^k} > \dfrac{p_t-\delta^{k+1}p_{t+k+1}}{1-\delta^{k+1}}, \forall k = 1, \ldots, T-t-1,$
$\forall t = 1, \ldots, T-1.$

(PROOF: see Appendix.)

This completes the preparations and allows us to state the main Theorem.

THEOREM 1. *The durable monopoly game has, for any fixed, finite T, a unique subgame perfect equilibrium outcome which is strict with respect to almost all players and which generically coincides with the only subgame perfect equilibrium outcome of the game. This unique outcome can be caracterized as follows:*

(i) In any period $t = 1, \ldots, T-1$ all consumers, who have not yet bought in any previous period $\tau < t$ and have a reservation value $v \in [0, 1]$ satisfying

$$v > \frac{p_t - \delta F_{t+1}(p_t)}{1-\delta}$$

will buy in period t, and all consumers, for whom the above strict inequality is reversed ($<$) will wait. In period $t = T$ all consumers, who have not yet bought and have a reservation value $v > p_T$ will buy, and all consumers $v < p_T$ will not buy.

(ii) The monopolist will in any period $t = 2, \ldots, T$ set a price $p_t = F_t(p_{t-1})$ and will in period $t = 1$ set the price $p_1 = \frac{1}{2}a_1$.

PROOF: The idea of the proof is to invoke Lemma 1 which generically and under strictness with respect to almost all players allows us to use $v_t(\sigma) = \sup \mathcal{V}_t(\sigma)$ as a state variable, and calculate the equilibrium by backward induction. Thus the proof is constructive and uniqueness is guaranteed by Lemma 1.

Let $t = T$. Clearly by subgame perfection applied to the last subgame of period T, any $v > p_T$, $v \in \mathcal{V}_T(\sigma)$, will buy and any $v < p_T$ will not buy. Thus the monopolist's problem in the first subgame of period T is to choose p_T such as to maximize

$$\pi_T = p_T \max\{0, v_T(\sigma) - p_T\}.$$

The first order condition of this problem reads for $v_T(\sigma) \geq p_T$

$$\begin{aligned} & v_T(\sigma) - 2p_T = 0 \\ \iff\ & p_T = \frac{1}{2}v_T(\sigma) = \frac{1}{2}a_T v_T(\sigma) \leq v_T(\sigma), \end{aligned}$$

with second order condition $-2 < 0$. Such a choice yields a profit of

$$\pi_T^*(v_T(\sigma)) = \frac{1}{4}v_T(\sigma)^2 = \frac{1}{4}a_T v_T(\sigma)^2 \geq 0$$

which is strictly larger than zero, if $v_T(\sigma) > 0$, and thus a unique best response. In period $t = T-1$ for all consumers $v \in \mathcal{V}_{T-1}(\sigma)$, who satisfy

$$v > \frac{p_{T-1} - \delta p_T}{1-\delta},$$

it is the unique best response to buy in period $t = T - 1$, and for all $v \in \mathcal{V}_{T-1}(\sigma)$, who satisfy

$$v < \frac{p_{T-1} - \delta p_T}{1 - \delta},$$

one must have) (Due to the rational expectation of Consumers $v_T(\sigma) = \frac{p_{T-1} - \delta \frac{v_T(\sigma)}{2}}{1-\delta} \Leftrightarrow v_T = \frac{p_{T-1}}{1-\frac{\delta}{2}}$. Consequently,

it is the unique best response to wait. ~~Consequently~~

$$v_T(\sigma) = \frac{p_{T-1} - \delta p_T}{1 - \delta} = \frac{p_{T-1} - \frac{\delta}{2} v_T(\sigma)}{1 - \delta} = \frac{p_{T-1}}{1 - \delta/2} > 0$$

which, in turn, implies ~~for~~

$$p_T = \frac{\frac{1}{2} p_{T-1}}{1 - \delta/2} = \frac{\frac{1}{2} a_T}{1 - \delta + \frac{\delta}{2} a_T} p_{T-1} = F_T(p_{T-1}),$$

verifies the monopolist's pricing rule for $t = T$ and $v_T(\sigma) > 0$.

The above implies the value function

$$\pi^*_T(p_{T-1}) = \frac{\frac{1}{4} p^2_{T-1}}{(1 - \delta/2)^2} = \frac{\frac{1}{4} a_T}{(1 - \delta + \frac{\delta}{2} a_T)^2} p^2_{T-1}.$$

still undefined (see page 2!)

Now assume that for some $t + 1 < T$ one has along the equilibrium path $p_{t+k}(\sigma) = F_{t+k}(p_{t+k-1}), \forall k = 1, \ldots, T - t$, and

$$\pi^*_{t+1}(p_t) = \frac{\frac{1}{4} a_{t+1}}{(1 - \delta + \frac{\delta}{2} a_{t+1})^2} p_t^2 = \frac{1}{4} a_{t+1} v_{t+1}(\sigma)^2$$

as the value function for the monopolist's profits from period $t + 1$ onwards. Consider the last subgame of period t, where consumers $v \in \mathcal{V}_t(\sigma)$ decide, given p_t. From the induction hypothesis $p_{t+1} = F_{t+1}(p_t)$ are rational expectations, and it follows that for any $v \in \mathcal{V}_t(\sigma)$, who satisfies

$$v > \frac{p_t - \delta F_{t+1}(p_t)}{1 - \delta}$$

it is the unique best response to buy in period t at p_t, because

$$\frac{p_t - \delta F_{t+1}(p_t)}{1 - \delta} > \frac{p_t - \delta^k p_{t+k}}{1 - \delta^k}$$

for all $k = 2, \ldots, T - t$, from Lemma 3, (ii), and the induction hypothesis. For any $v \in \mathcal{V}_t(\sigma)$, who satisfies

$$v < \frac{p_t - \delta F_{t+1}(p_t)}{1 - \delta},$$

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HH not to buy at price p_t

it is a best ~~choice to wait~~ ~~for at least $t+1$~~. Thus

$$v_{t+1}(\sigma) = \frac{p_t - \delta F_{t+1}(p_t)}{1-\delta} = \frac{p_t}{1-\delta+\frac{\delta}{2}a_{t+1}}.$$

Consequently, the monopolist's problem in the first subgame of period t is to choose p_t such as to maximize

$$\begin{aligned}\pi_t &= p_t[v_t(\sigma) - \frac{p_t}{1-\delta+\frac{\delta}{2}a_{t+1}}] + \frac{\frac{\rho}{4}a_{t+1}}{(1-\delta+\frac{\delta}{2}a_{t+1})^2}p_t^2 = \\ &= p_t v_t(\sigma) - \frac{1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_{t+1}}{(1-\delta+\frac{\delta}{2}a_{t+1})^2}p_t^2 = \\ &= p_t v_t(\sigma) - \frac{p_t^2}{a_t},\end{aligned}$$

if this quantity is larger than what can be obtained from $p_t \geq v_t(\sigma)$. The first order condition for this problem is

$$v_t(\sigma) - \frac{2p_t}{a_t} = 0$$

with second order condition $-2/a_t < 0$ by Lemma 2, (iii). Such a choice yields a profit of

$$\pi_t^*(v_t(\sigma)) = \frac{1}{4}a_t v_t(\sigma)^2$$

and a price p_t of

$$p_t = \frac{1}{2}a_t v_t(\sigma) < v_t(\sigma)$$

as the unique best response, by Lemma 2, (iii). In order to see that the monopolist does not have an incentive to set $p_t \geq v_t(\sigma)$, observe that, if he does, then $v_{t+1}(\sigma) = v_t(\sigma)$ implies

$$\pi_t(v_t(\sigma)) = \rho\pi_{t+1}^*(v_{t+1}(\sigma)) = \rho\pi_{t+1}^*(v_t(\sigma)) = \frac{\rho}{4}a_{t+1}v_t(\sigma)^2$$

such that

$$\begin{aligned}&\frac{\rho}{4}a_{t+1}v_t(\sigma)^2 < \frac{(1-\delta+\frac{\delta}{2}a_{t+1})^2 v_t(\sigma)^2}{4(1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_{t+1})} \quad \checkmark\\ \Longleftrightarrow\ &\rho a_{t+1}[1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_{t+1}] < (1-\delta+\frac{\delta}{2}a_{t+1})^2\\ \Longleftrightarrow\ &\rho(1-\delta)a_{t+1} + \rho(\frac{\delta}{2}-\frac{\rho}{4})a_{t+1}^2 < (1-\delta)^2 + \delta(1-\delta)a_{t+1} + \frac{\delta^2}{4}a_{t+1}^2\\ \Longleftrightarrow\ &0 < (1-\delta)^2 + (\delta-\rho)(1-\delta)a_{t+1} + (\frac{\delta^2}{4} - \frac{\rho\delta}{2} + \frac{\rho^2}{4})a_{t+1}^2\\ \Longleftrightarrow\ &0 < [1-\delta+(\frac{\delta}{2}-\frac{\rho}{2})a_{t+1}]^2,\end{aligned}$$

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by Lemma 2, (i). By the same argument as above, the consumers' optimal behavior in the last subgame of period $t-1$ yields

$$v_t(\sigma) = \frac{p_{t-1} - \frac{\delta}{2} a_t v_t(\sigma)}{1-\delta} = \frac{p_{t-1}}{1-\delta+\frac{\delta}{2}a_t}$$

such that

$$\pi_t^*(p_{t-1}) = \frac{\frac{1}{4}a_t}{(1-\delta+\frac{\delta}{2}a_t)^2} p_{t-1}^2$$

verifies the value function for period t. Also this verifies

$$p_t = \frac{1}{2} a_t v_t(\sigma) = \frac{\frac{1}{2}a_t}{1-\delta+\frac{\delta}{2}a_t} p_{t-1} = F_t(p_{t-1}).$$

This completes the induction argument. All that remains to be shown is $p_t(\sigma) \leq v_{t-1}(\sigma)$. But this follows from

$$v_t(\sigma) = \frac{p_{t-1} - \delta p_t}{1-\delta} \leq \frac{p_{t-2} - \delta p_{t-1}}{1-\delta} = v_{t-1}(\sigma),$$

$p_t = F_t(p_{t-1})$, and $p_{t-1} = F_{t-1}(p_{t-2})$, and Lemma 3, (i). This completes the verification of the subgame perfect equilibrium strategies.

Observe that the equilibrium constructed above is one, where all players play their unique best responses against the equilibrium strategies of the other players except for the finitely many consumers, who satisfy $v = v_t(\sigma)$, for some $t = 1, \ldots, T$. Since finitely many points in the unit interval have measure zero, this equilibrium is strict with respect to almost all players, and thus its outcome (up to the behavior of the finitely many indifferent consumers) is unique by Lemma 1, as required by the Theorem. ∎

The equilibrium constructed in the proof of Theorem 1 displays the following recursive properties:

$$(1)\quad p_t(\sigma) = \frac{1}{2}a_t v_t(\sigma) = F_t(p_{t-1}(\sigma)), \quad p_1(\sigma) = \frac{1}{2}a_1;$$

$$(2)\quad v_t(\sigma) = \frac{p_{t-1}}{1-\delta+\frac{\delta}{2}a_t} = \frac{(1-\delta+\frac{\delta}{2}a_t)v_{t-1}(\sigma)}{2(1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_t)}, \quad v_1(\sigma) = 1;$$

$$(3)\quad \pi_t^*(\sigma) = \frac{1}{4}a_t v_t(\sigma)^2 = \frac{\frac{1}{4}a_t \pi_{t-1}^*(\sigma)}{1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_t}, \quad \pi_1^*(\sigma) = \frac{a_1}{4}.$$

Once the sequence of coefficients $(a_t)_{t=1}^T$ is calculated, equations (1)-(3) allow a simple calculation of the three important variables of the game:

prices, active consumers, and profits (along the equilibrium path). Since the a_t's only depend on T, it is possible to write $p_t(T)$, $v_t(T)$, and $\pi_t^*(T)$ instead of using σ as the argument. Observe from (1) that prices always decrease over time, because

$$\frac{1}{2}a_t < 1 - \delta + \frac{\delta}{2}a_t \iff a_t < 2,$$

from Lemma 2, (iii).

Comparative statics can also be easily done with the help of equation (1). As intution suggests, a more patient monopolist will set higher prices.

PROPOSITION 1. *The more patient the monopolist is, the higher are the prices, i.e. $\partial p_t(T)/\partial \rho > 0$, for all $t = 1, \ldots, T$.*

PROOF: First observe that

$$\frac{\partial a_t}{\partial \rho} = \frac{1 - \delta + \frac{\delta}{2}a_{t+1}}{(1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a_{t+1})^2}[\frac{1}{4}(1 - \delta + \frac{\delta}{2}a_{t+1})a_{t+1} +$$
$$+ \{\frac{\delta}{2}(1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a_{t+1}) + \frac{\rho}{4}(1 - \delta)\}\frac{\partial a_{t+1}}{\partial \rho}],$$

Then $\frac{da_T}{d\rho} = 0$ [illegible] drives

such that by Lemma 2, $\partial a_{t+1}/\partial \rho \geq 0$ implies $\partial a_t/\partial \rho > 0$ implying $\partial a_t/\partial \rho > 0, \forall t = 1, \ldots, T-1$. Now from (1)

$$\frac{\partial p_t}{\partial \rho} = \frac{(1-\delta)p_{t-1}}{2(1 - \delta + \frac{\delta}{2}a_t)^2}\frac{\partial a_t}{\partial \rho} + \frac{a_t}{2(1 - \delta + \frac{\delta}{2}a_t)}\frac{\partial p_{t-1}}{\partial \rho},$$

such that $\partial p_1/\partial \rho = \frac{1}{2}\partial a_1/\partial \rho > 0$ implies $\partial p_t/\partial \rho > 0$, for all $t = 1, \ldots, T$, as required. ∎

Conclusions on the reaction of prices to the time preference parameter δ of consumers are not so easy to have, because δ enters the definition of F_t directly rather than only through a_t and p_{t-1}. Such conclusions will be easier to derive from the limiting solution studied below.

Theorem 1 reveals, why the traditional wisdom that the monopolist has to price competitively, because his own agents of different periods compete against each other, does not hold in the present setting: The Bertrand-type competition of the monopolist's agents across periods is an artefact of stating the problem in an infinite horizon in the first place, rather than considering the limit of the solution for finite T (where T is the number of subperiods in a time interval of fixed length), when the grid on a finite time interval becomes infinitely fine, as it was the

Meines Erachtens ist die Coase-Conjecture kein Artefakt von $T = \infty$, sondern nur der asymmetrischen Zeitpräferenz, nämlich für geduldige Käufer und einen sehr ungeduldigen Verkäufer ($\delta \to 1$, $\rho \to 0$)

case in the original Coase-Conjecture. When the limiting solution for finite T is studied, $T \to \infty$, agents do not compete as fiercly anymore, because the possibilities of a precommitment increase over time: The agent in the last period has, by subgame perfection, no other option than behaving as a one-shot monopolist against the remaining set of still active consumers. Because rational expectations obtain in equilibrium, the monopolist's agent of period $T-1$ knows that the agent of period T is commited to do so, and can, therefore, price-discriminate. This structure unravels backwards. The consequences of this are that, as there are more subperiods, the monopolist's overall profit more and more resembles the profit he would gain, if he would wait for the last period, where he behaves monopolistically.

Kleiner Einfluss konvergieren alle Preise gegen denselben Wert wenn $\delta \to 1$, das ist der Fall, wenn die Produkte der verschiedenen Perioden homogen werden, so dass man hier von Rotierend-Bestellbarkeit sprechen kann

The next proposition pins down, what these remarks suggest. Let now the time horizon be normalized to 1 and consider the durable monopoly game in T subperiods of length $\Delta = 1/T$. The appropriate discount factors are now given by

$$\delta = e^{-r\Delta}, \quad \rho = e^{-R\Delta}, \quad r > 0,\ R > 0.$$

This means we restrict the analysis essentially to one parameter, i.e. we study the limit $T \to \infty$ for $\delta, \rho \to 1$.

Although players do almost not discount from one subperiod to the next, they do discount over the entire time horizon from $t = 0$ to $t = 1$, when $\Delta \to 0$.

Proposition 2 (Anti-Coase-Conjecture). *If $\Delta \to 0$, i.e. $T \to \infty$, then $a_{t\Delta} \longrightarrow a(t) = \exp\{Rt - R\}$, $\forall t \in (0, 1]$, and*

$$\pi^*(0) = lim_{\Delta \to 0}\pi_1^*(1/\Delta) = e^{-R}/4 > 0.$$

Proof: Rewrite the definition of the sequence of coefficients $(a_t)_{t=1}^{T}$ as

$$a_{1-\tau\Delta} = \frac{(1 - e^{-r\Delta} + \frac{1}{2}e^{-r\Delta}a_{1-\tau\Delta+\Delta})^2}{1 - e^{-r\Delta} + (\frac{1}{2}e^{-r\Delta} - \frac{1}{4}e^{-R\Delta})a_{1-\tau\Delta+\Delta}}, \quad a_1 = 1,$$

for all $\tau = 1, \ldots, T-1$. Then

$$\frac{a_{1-\tau\Delta+\Delta} - \cancel{a_{1-\tau\Delta}}}{\Delta} = \frac{1}{\Delta}[1 - e^{-r\Delta} + (\frac{e^{-r\Delta}}{2} - \frac{e^{-R\Delta}}{4})a_{1-\tau\Delta+\Delta}]^{-1} \times$$

$$\times [(1 - e^{-r\Delta} + (\frac{e^{-r\Delta}}{2} - \frac{e^{-R\Delta}}{4})a_{1-\tau\Delta+\Delta})a_{1-\tau\Delta+\Delta} -$$

$$- (1 - e^{-r\Delta} + \frac{e^{-r\Delta}}{2}a_{1-\tau\Delta+\Delta})^2].$$

$H / a_{1-\tau\Delta}$

V

Taking limits, using L'Hospital, yields

$$\lim_{\Delta\to 0}\frac{(1-e^{-r\Delta})a+(\frac{e^{-r\Delta}}{2}-\frac{e^{-R\Delta}}{4})a^2-(1-e^{-r\Delta}+\frac{e^{-r\Delta}}{2}a)^2}{\Delta(1-e^{-r\Delta}+(\frac{e^{-r\Delta}}{2}-\frac{e^{-R\Delta}}{4})a)}=$$

$$=\lim_{\Delta\to 0}[re^{-r\Delta}a-(\frac{re^{-r\Delta}}{2}-\frac{Re^{-R\Delta}}{4})a^2-2(1-e^{-r\Delta}+$$

$$+\frac{e^{-r\Delta}}{2}a)re^{-r\Delta}(1-\frac{a}{2})][1-e^{-r\Delta}+(\frac{e^{-r\Delta}}{2}-\frac{e^{-R\Delta}}{4})a+$$

$$+\Delta(re^{-r\Delta}-(\frac{re^{-r\Delta}}{2}-\frac{Re^{-R\Delta}}{4})a)]^{-1}=Ra\,, \quad \checkmark$$

such that $\dot a(t)=Ra(t)$, $a(1)=1$. This differential equation has the solution $a(t)=\exp\{Rt-R\}$. This implies from (3) that $\bar\pi^*(0)=\lim_{\Delta\to 0}\pi_1^*(1/\Delta)=a(0)/4=e^{-R}/4$, as required. ∎ ✓

It is of some interest to calculate the limiting paths for prices and the sets of active consumers. From (1) the pricing equation can be written

HRΔ-R

$$p_{t\Delta+\Delta}=\frac{\frac{1}{2}a_{t\Delta+\Delta}}{1-e^{-r\Delta}+\frac{1}{2}e^{-r\Delta}a_{t\Delta+\Delta}}p_{t\Delta},\quad p_\Delta=\frac{e^{-R}}{2}.$$

1 Δ

l'Hospital

This yields from

$$\lim_{\Delta\to 0}\boxed{\frac{\frac{a}{2}-1+e^{-r\Delta}-\frac{e^{-r\Delta}}{2}a}{\Delta(1-e^{-r\Delta}+\frac{e^{-r\Delta}}{2}a)}}=\lim_{\Delta\to 0}\frac{-re^{-r\Delta}+\frac{re^{-r\Delta}}{2}a}{1-e^{-r\Delta}+\frac{e^{-r\Delta}}{2}a}=$$
$$=r(1-2/a)<0,\quad a\in(0,2),$$

$\left[\frac{p_{t\Delta+\Delta}-1}{\Delta}\right]$?

$T+\Delta re^{-r\Delta}(1-\frac{a}{2})$

the differential equation $\dot p(t)=r[1-2\exp\{R-Rt\}]p(t)$, with initial condition $p(0)=\exp\{-R\}/2$, which has the solution

$$(4)\qquad p(t)=\frac{1}{2}\exp\{rt-\frac{2r}{R}e^{R}(1-e^{-Rt})-R\}.$$

This provides for the limiting case an interesting extra part to Proposition 1 for consumers: The more patient consumers are (the smaller r is), the *higher* prices are, because $Rt<2e^{R}(1-e^{-Rt})$, $\forall t\in(0,1]$. Thus consumers, who do not mind waiting, make it easier for the monopolist to let prices drop more slowly.

By an analogous procedure as for prices, one obtains from (2)

$$v_{t\Delta+\Delta}=\frac{(1-e^{-r\Delta}+\frac{1}{2}e^{-r\Delta}a_{t\Delta+\Delta})v_{t\Delta}}{2(1-e^{-r\Delta}+(e^{-r\Delta}/2-e^{-R\Delta}/4)a_{t\Delta+\Delta})},\quad v_\Delta=1,\quad\checkmark$$

$$\lim_{\Delta\to 0}\frac{\frac{1}{2}e^{-r\Delta}-\frac{1}{2}-\frac{1}{4}e^{-r\Delta}a+\frac{1}{4}e^{-R\Delta}a}{\underbrace{\Delta(1-e^{-r\Delta}+(e^{-r\Delta}/2-e^{-R\Delta}/4)a)}_{?}}=r-R-2r/a<-R,\quad\checkmark$$

l'Hospital

if $a \in (0, 2)$, such that $\dot{v}(t) = [r - R - 2r\exp\{R - Rt\}]v(t)$, with initial condition $v(0) = 1$ yields

$$v(t) = \exp\{(r - R)t - \frac{2r}{R}e^{R}(1 - e^{-Rt})\}. \tag{5}$$

Note that $v(t)$ will always satisfy $v(t) \in (0, 1)$, $\forall t \in (0, 1]$, because $v(0) = 1$ and $\dot{v}(t) < 0$. Since $v(1) > 0$ there will always remain a non-empty set of unsatisfied consumers, and - depending on r and R - this set may even be larger than in the one-shot game (this will for example be the case, if $\ln(2) > R$ and $r < [2(e^{R} - 1) - R]^{-1}[R(\ln(2) - R)]$ holds).

Appendix

Proof of Lemma 2: Let $t = T$; then $a_T = 1$, such that $1 - \delta + \delta/2 = 1 - \delta/2 > \rho/2 > \rho/4$ and $0 < a_T = 1 < 2$ hold. Then, in order to apply induction, assume that

$$\begin{aligned}
(i')\quad & 1 - \delta + \frac{\delta}{2}a_{t+1} > \frac{\rho}{4}a_{t+1},\\
(ii')\quad & 1 - \delta + \frac{\delta}{2}a_{t+1} > \frac{\rho}{2}a_{t+1},\\
(iii')\quad & 0 < a_{t+1} < 2.
\end{aligned}$$

Then (i') implies $a_t > 0$. From

$$\begin{aligned}
\frac{\partial}{\partial a}\frac{(1 - \delta + \frac{\delta}{2}a)^2}{1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a} &= [(1 - \delta + \frac{\delta}{2}a)[\frac{\delta}{2}(1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a) +\\
&+ \frac{\rho}{4}(1 - \delta)]][1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a]^{-2}\Big|_{0<a<2} >\\
&> \frac{(1 - \delta + \frac{\delta}{2}a)[\frac{\delta}{4}(1 - \frac{\rho}{2})a + \frac{\rho}{4}(1 - \delta)]}{[1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a]^2} > 0,
\end{aligned}$$

(which follows by using $\frac{\delta}{2}(1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a) > \frac{\delta}{4}(1 - \frac{\rho}{2})a \iff 2 > a > 0$) it follows that (iii') implies

$$a_t < \frac{1}{1 - \rho/2} < 2,$$

where the second inequality follows from $\rho < 1$. Thus (i') and (iii') imply (iii).

Finally observe that

$$\frac{\partial}{\partial \delta}[1 - \delta + (\frac{\delta}{2} - \frac{\rho}{4})a_t] = \frac{1}{2}a_t - 1 < 0,$$

This implies

$$2^k \prod_{i=0}^{k-1} \frac{1-\delta+\frac{\delta}{2}a_{t+k-i}}{a_{t+k-i}} > \frac{2(1-\delta^k)+\delta^k a_{t+k}}{a_{t+k}} =$$

$$= \frac{2(1-\delta^k)[1-\delta+(\frac{\delta}{2}-\frac{\rho}{4})a_{t+k+1}]+\delta^k[1-\delta+\frac{\delta}{2}a_{t+k+1}]^2}{[1-\delta+\frac{\delta}{2}a_{t+k+1}]^2} =$$

$$= [(1-\delta+\frac{\delta}{2}a_{t+k+1})[2(1-\delta^k)+\delta^k(1-\delta+\frac{\delta}{2}a_{t+k+1})]-$$

$$-\frac{\rho}{2}(1-\delta^k)a_{t+k+1}][1-\delta+\frac{\delta}{2}a_{t+k+1}]^{-2} \geq$$

$$\geq \frac{1-\delta^{k+1}+\frac{\delta^{k+1}}{2}a_{t+k+1}}{1-\delta+\frac{\delta}{2}a_{t+k+1}},$$

because the second (weak) inequality is equivalent to

$$2(1-\delta^k)+\delta^k(1-\delta+\frac{\delta}{2}a_{t+k+1})-\frac{\frac{\rho}{2}(1-\delta^k)a_{t+k+1}}{1-\delta+\frac{\delta}{2}a_{t+k+1}} \geq$$

$$\geq 1-\delta^{k+1}+\frac{\delta^{k+1}}{2}a_{t+k+1} \iff$$

$$1-\delta+\frac{\delta}{2}a_{t+k+1} \geq \frac{\rho}{2}a_{t+k+1}$$

which, again, follows from Lemma 2, (ii). ∎

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