# Cooperation in Repeated Prisoner's Dilemma with Outside Options*

by

Takako Fujiwara-Greve†
Department of Economics, Keio University
2-15-45 Mita, Minato-ku, Tokyo 108-8345 JAPAN

and

Yosuke Yasuda‡
National Graduate Institute for Policy Studies (GRIPS)
7-22-1 Roppongi, Minato-ku, Tokyo 106-8677 JAPAN

June 20, 2009

**Abstract**: In many repeated interactions, repetition is not guaranteed but instead must be agreed upon. We formulate a model of voluntary repetition by introducing outside options to a repeated Prisoner's Dilemma and investigate how the *structure* of outside options affects the sustainability of mutual cooperation. When the outside option is deterministic and greater than the value of mutual defection, the lower bound of the discount factors that sustain repeated cooperation is greater than the one for ordinary repeated Prisoner's Dilemma, making cooperation more difficult. However, stochastic outside options with the same mean may reduce the lower bound of discount factors as compared to the deterministic case. This is possible when the stochasticity of the options increases the value of the cooperation phase more than the value of the punishment phase. Necessary and sufficient conditions for this positive effect are given under various option structures.

Key words: outside option, repeated Prisoner's Dilemma, cooperation, perturbation.
JEL classification number: C 73

*We are grateful to Satoru Takahashi, Takashi Shimizu, Henrich R. Greve, and seminar participants at KIER, Kyoto University, European Meeting of the Econometric Society (Milan), and Japanese Economic Association meeting at Kinki University for useful comments on earlier drafts. Usual disclaimer applies.

†E-mail: takakofg@econ.keio.ac.jp

‡E-mail: yyasuda@grips.ac.jp

# 1. INTRODUCTION

In many repeated interactions, repetition is not guaranteed but instead must be agreed upon. Workers can quit, customers can walk away, and couples can break up. If it is possible to strategically exit from a repeated interaction, the ordinary repeated-game framework no longer applies. Ordinary repeated games assume that the same set of players play the same stage game repeatedly for a fixed (possibly infinite) length of time. Therefore no player has a choice to exit from the game. At the other extreme, random matching games[1] assume that in every period a player is randomly matched with a new partner. Therefore no player has a choice to continue the game with the same partner. However, many economic situations are in an intermediate case where players can play a game repeatedly, but they can also terminate the interaction. There is a growing literature of these "endogenously repeated" games.

In this literature, three issues have been mainly analyzed. First, under complete information, ordinary trigger strategies do not constitute an equilibrium since cooperation from the beginning of a relationship is vulnerable to defection and running away. Instead, gradual cooperation or trust-building strategy becomes an equilibrium. (Datta, 1996, Kranton, 1996a, Fujiwara-Greve, 2002, and Fujiwara-Greve and Okuno-Fujiwara, 2009.) Second, gradual cooperation is also useful in incomplete information models to sort out the types of players. (Ghosh and Ray, 1996, Kranton, 1996a,b, Watson, 2002, and Furusawa and Kawakami, 2008.) Third, a modified folk theorem holds with appropriate lower bounds of the equilibrium payoffs. (Yasuda, 2007.)

We add a new angle to the analysis of the endogenously repeated games by looking at the interaction between in-game behavior and what a player may receive outside of the game. In game theory, often the outside structure of a game is fixed and the analysis is focused on in-game strategic outcomes given the outside structure.[2] By contrast in other research fields such as search theory and operations research, the main interest lies in the effect of outside structural changes on individual behavior/decision-making, but there is no strategic interaction among decision-makers. In this paper we consider strategic interaction of two players under varying outside structures of the game.

Specifically, we examine variants of the repeated Prisoner's Dilemma from which players can exit by taking an outside option and investigate effects of outside option

[1]See for example, Kandori (1992), Ellison, (1994), and Okuno-Fujiwara and Postlewaite (1995).

[2]However, there are some papers that perturb the game structure to see the effect on in-game behavior. See the discussion below.

| P1 \ P2 | C | D |
|---|---|---|
| C | 7, 7 | 0, 9 |
| D | 9, 0 | 1, 1 |

Table 1: An Example

structures on the sustainability of cooperation. It turns out that the "locked-in" feature of ordinary repeated game is a very strong cooperation enforcement system. The existence of a relevant outside option (greater than the in-game punishment payoff) increases the necessary level of discount factor to sustain cooperation as compared to the one in ordinary repeated games, and in some cases for any discount factor cooperation is not possible. However, within the outside option model, the relative difficulty of repeated cooperation is dependent on the structure of outside options. In particular, if the option values are uncertain, in some cases it is easier to sustain repeated cooperation than when they are certain. Therefore, perturbation of outside options is not always bad for cooperation.

Let us give an example to explain the logic. In each period, as long as the two players are in the game, they play the Prisoner's Dilemma of Table 1. After playing the Prisoner's Dilemma, an outside option is available to Player 1. Player 2 has no such option. The game repeats (Prisoner's Dilemma and then the outside option to exit) as long as Player 1 does not take the outside option. Suppose that in any period the available outside option is the same, and it gives a stationary sequence of payoff $\{6, 6, \ldots\}$ to Player 1 after exit. Player 2's payoff after Player 1 ends the game is normalized to be zero.

Note that if the game is an ordinary repeated game without the outside option, the infinitely repeated cooperation $(C, C), (C, C), \ldots$ (which we call *the eternal cooperation*) is sustainable by the grim trigger strategy if

$$\frac{7}{1-\delta} \geqq 9 + \delta \frac{1}{1-\delta} \iff \delta \geqq \frac{1}{4}.$$

However, if the outside option is available, Player 1 can choose D and take the option $\{6, 6, \ldots\}$. Therefore, Player 1 may not follow the eternal cooperation $(C, C), (C, C), \ldots$ even if $\delta$ is not so small. For example, when $\delta = 0.6$,

$$\frac{7}{1-\delta} = 17.5 < 18 = 9 + \delta \frac{6}{1-\delta}.$$

This illustrates that the existence of an outside option greater than the in-game punishment payoff creates difficulty in achieving cooperation, in the sense that the range of discount factors that sustain repeated cooperation shrinks.

Next, suppose that Player 1 has two possible outside options of the form $\{6+\epsilon, 6+\epsilon, \ldots\}$ and $\{6-\epsilon, 6-\epsilon, \ldots\}$ (where $\epsilon > 0$), and these arrive with equal probability at the end of each period. The average outside option is 6. When $\epsilon$ is small (i.e., less than 1), then there is no point of taking any of the outside options if players are to repeat $(C, C)$. When $\epsilon$ is large enough, however, the better outside option exceeds the payoff from the repeated $(C, C)$ so that the infinitely repeated cooperation becomes impossible for any $\delta$. However, Player 1 may cooperate until she receives the better option. Let us compute the total expected discounted payoff of cooperation until the better option arrives. Let $V$ be the continuation value at the end of a period, before an option realizes. Then the total expected payoff of repeating $(C, C)$ until $\{6+\epsilon, 6+\epsilon, \ldots\}$ arrives is of the form $7 + \delta V$, where the continuation value $V$ satisfies the following recursive equation.

$$V = \frac{1}{2} \cdot \frac{6+\epsilon}{1-\delta} + \frac{1}{2}(7 + \delta V).$$

For example, when $\epsilon = 1.5$ and $\delta = 0.6$, then $V \approx 18.39$, and the value of the cooperation is increased to $7 + \delta V \approx 18.03 > 17.5 = 7/(1-\delta)$.

The value of a one-shot deviation also needs to be checked more carefully. The optimal exit strategy for Player 1 is either to exit immediately by taking any option or to wait for $\{6+\epsilon, 6+\epsilon, \ldots\}$. If she deviates and then waits for the good option while suffering from the punishment payoff of 1 in the stage game, the total expected payoff is of the form $9 + \delta W$, where the continuation value $W$ satisfies

$$W = \frac{1}{2} \cdot \frac{6+\epsilon}{1-\delta} + \frac{1}{2}(1 + \delta W).$$

Thus $9 + \delta W \approx 17.46$ for $\epsilon = 1.5$ and $\delta = 0.6$. If Player 1 defects and then exits immediately by taking any option, the expected payoff is $9 + \delta \frac{6}{1-\delta} = 18$ as before. Therefore, in this example, it is optimal to exit immediately after a deviation. However, $7 + \delta V > 18 = 9 + \delta \frac{6}{1-\delta}$ implies that Player 1 with $\delta = 0.6$ cooperates on the play path, until the better outside option arrives. We call this play path *stochastic cooperation.* It is better than no cooperation, which is the case if the outside option was deterministic.

The above example shows that deterministic or stochastic structure of outside options makes a difference in sustaining cooperation for mid-range discount factors. In addition, given a discount factor, we can investigate how the spread $\epsilon$ of the outside options affects the sustainability of cooperation. In this example, when $\epsilon$ is small, no cooperation is possible, just like in the deterministic case. As $\epsilon$ increases, the value of cooperation while waiting for the good option increases so that stochastic cooperation becomes an equilibrium behavior.[3] This can be generalized for a mid-range of $\delta$. Therefore the perturbation of outside options may enhance cooperation.

Moreover, we can vary the probability of the binary options and show qualitatively same results: when the better outside option exceeds the cooperation payoff, the stochastic cooperation becomes an equilibrium for mid-range discount factors. Hence, as the probability of the attractive option decreases, the stochastic cooperation lasts longer on average, and the play path becomes almost the eternal cooperation.

There are a few papers which incorporate perturbations into ordinary repeated games. Rotemberg and Saloner (1986) perturb payoffs of the stage game, while Baye and Jansen (1996) and Dal Bó (2007) perturb the discount factor. In these models the optimal (eternal) cooperation levels are shown to be lower than the one in the absence of perturbation. In this sense, the perturbations are bad for cooperation. Although they did not investigate the lower bound of the discount factors by fixing a level of cooperation, it would be greater than the one under no perturbation. This is clarified in Yasuda and Fujiwara-Greve (2009).

The key is that the players are locked in the game forever in the infinitely repeated games. Therefore, when the perturbation creates difficulty to cooperate (a high deviation payoff or a low value of the discount factor), the players need to play a non-cooperative action in that period, which *reduces* the on-path payoff, i.e., the incentive to follow the equilibrium strategy. Therefore the players need to be more patient than in the deterministic case.[4]

By contrast, in our outside option model, Player 1 can choose between playing the game forever and stopping. Thus, when the perturbation creates a difficulty to cooperate (a high outside option), it does not mean that Player 1 must endure the low payoff of a non-cooperative action. The difficulty to cooperate means that stopping the game is more beneficial, and hence she can take that option to *increase* the on-path payoff, i.e., the incentive to follow the equilibrium strategy. Therefore lower discount

[3]As $\epsilon$ increases more, e.g., $\epsilon = 2.5$, then $9 + \delta W > 9 + \delta \frac{6}{1-\delta}$ so that after defection, Player 1 wants to wait for the better option. However, $7 + \delta V > 9 + \delta W$ holds so that the stochastic cooperation continues to be an equilibrium behavior.

[4]A similar argument is noted in Mailath and Samuelson (2006), p.176-177.

factors are sufficient to sustain the equilibrium as compared to the deterministic case.

In summary, we have shown that there are perturbations that can increase the value of repeated cooperation, and this occurs naturally in the context of outside options in the endogenously repeated game.

The outline of the paper is as follows. In Section 2, we formulate the basic one-sided outside option model. In Section 3, we show that the existence of a deterministic outside option makes it harder to cooperate than in the ordinary repeated Prisoner's Dilemma. In Section 4, we consider a one-sided stochastic outside option model and show that stochastic outside options may enhance cooperation, as compared to the deterministic options. In Section 5, we give two extensions. One is a two-sided outside option model, in which the effect of perturbation is weakened because a player may end up with a bad option when the opponent receives a good option and terminates the game. This reduces the values of both cooperation and punishment phases and weakens the perturbation effect. The other is a continuous distribution of (one-sided) outside options. The results are essentially the same as the ones of binary distributions. Section 6 gives concluding remarks.

## 2. A ONE-SIDED OUTSIDE OPTION MODEL

Consider a two-player dynamic game as follows. Time is discrete and denoted as $t = 1, 2, \ldots$ but the game continues endogenously. At the beginning of period $t = 1, 2, \ldots$ as long as the game continues, two players, called Player 1 and Player 2, simultaneously choose one of the actions from the set $\{C, D\}$ of the Prisoner's Dilemma. The action $C$ is interpreted as a cooperative action and the action $D$ is interpreted as a defective action. We denote the symmetric payoffs associated with each action profile as[5]: $u(C, C) = c$, $u(C, D) = \ell$, $u(D, C) = g$, $u(D, D) = d$ with the ordering $g > c > d > \ell$ and $2c > g + \ell$. See Table 2. The latter inequality implies that $(C, C)$ is efficient among correlated action profiles.

After observing this period's action profile, Player 1 can choose whether to take an outside option and thus terminate the game or not. The game continues to the next period if and only if Player 1 does not take an outside option. We assume that all actions are observable to the players. Therefore, in period $t \geqq 2$, players can base their actions on the history of past action profiles. The outline of the dynamic game is depicted in Figure 1.

[5]The first coordinate is the player's own action, and the second coordinate is the opponent's action.

| P1 \ P2 | C | D |
|---|---|---|
| C | $c$, $c$ | $\ell$, $g$ |
| D | $g$, $\ell$ | $d$, $d$ |

Table 2: General Prisoner's Dilemma

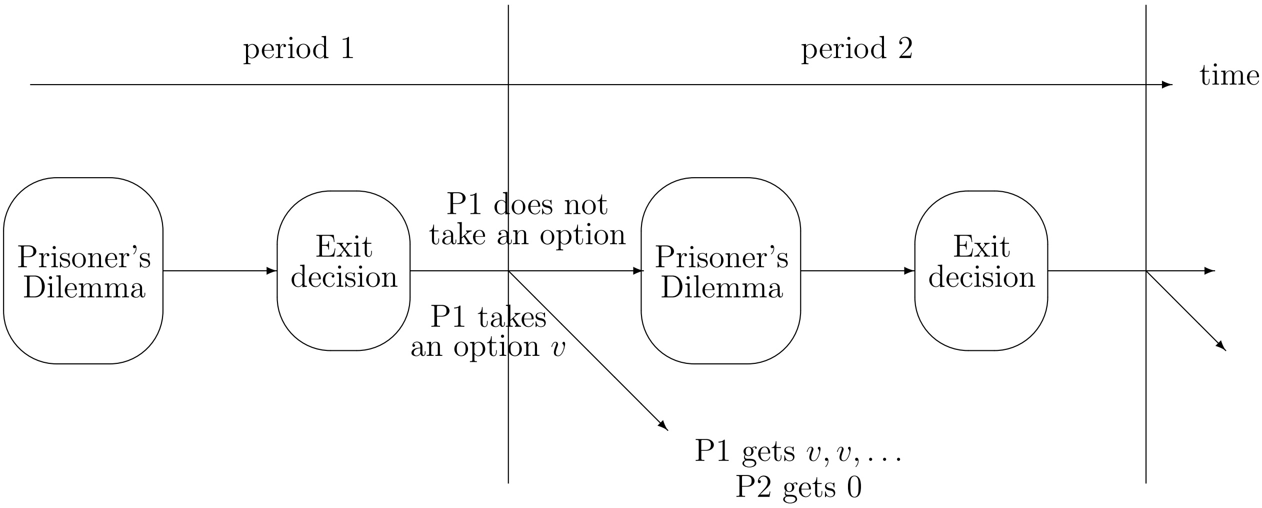



Figure 1: Outline of the Dynamic Game

As the basic setup, let an outside option be a deterministic, stationary stream of payoffs $\{v, v, \ldots\}$, such that $c > v > d$.[6] One can alternatively assume that an outside option is a one-shot payoff of the form $v/(1-\delta)$, where $\delta$ is the common discount factor. Other outside option structures (stochastic, two-sided) are discussed in later sections.

Player 2 receives payoff only from the Prisoner's Dilemma as long as the game continues and Player 2 does not have the ability to end the game, as in the ordinary repeated games. Let us also assume that $d \geqq 0$ which implies that Player 2's "outside payoff" 0 is not better than the payoff from $(D, D)$. This simplifies our analysis by making Player 2's deviation not relevant. (To be precise, the qualitative result does not change as long as Player 2's outside payoff is not greater than $v$.)

There are many economic situations that fit into this model. For example, we can interpret the model as a buyer-seller model such that Player 1 is a buyer, Player 2 is a seller, $C$ is an honest action in transactions and $D$ is a dishonest action. We can also

[6]If $v > c$, then $(C, C)$ cannot be played at all, and if $v < d$, then the outside option is never taken so that the game essentially reduces to an ordinary repeated Prisoner's Dilemma.

interpret the model as an employment relationship such that Player 1 is a worker and Player 2 is a firm.

We assume that both players maximize the discounted sum[7] of the payoff stream with a common discount factor $\delta \in (0,1)$. For example, if Player 1 takes the outside option at the end of $T$-th period, her total payoff is

$$\sum_{t=1}^{T} \delta^{t-1} u(a(t)) + \delta^{T} \frac{v}{1-\delta},$$

while Player 2's total payoff is

$$\sum_{t=1}^{T} \delta^{t-1} u(a(t)),$$

where $a(t)$ is the action profile in $t$-th period of the repeated Prisoner's Dilemma.

As the equilibrium concept, we use subgame perfect equilibrium (SPE henceforth). The game is of complete information.

LEMMA 1. *The following strategy combination is a SPE for any* $v \in (d,c)$ *and any* $\delta \in (0,1)$*: In any period of the game, Player 1 and Player 2 play D and Player 1 takes the outside option, regardless of the history.*

Proof: Given the strategy combination, both players get $d$ in every period if they are in the game. Therefore, at any exit decision node, taking the outside option is optimal for Player 1 since $v > d$. Given Player 1's exit strategy, it is optimal for both players to play myopically in every period (if they are still together). □

Notice that Player 1 can guarantee herself the total payoff of $d + \delta \frac{v}{1-\delta}$ against any strategy of Player 2 by choosing D and exiting immediately, while Player 2 can guarantee $d + 0$ against any strategy of Player 1. Since the above SPE (called the "myopic SPE" henceforth) achieves exactly these payoffs, it is the maximal equilibrium punishment.

We investigate the range of $\delta$ in which repeated mutual cooperation of $(C,C)$ is sustained as long as possible. If the maximal equilibrium punishment does not sustain the on-path action profile, no other punishment would, by the same logic as the optimal penal code in Abreu (1988). Hence it is sufficient to consider the myopic SPE

[7]Alternatively one can assume that the players maximize the average payoffs without changing the qualitative results.

as the punishment. Therefore, in general we consider the following type of strategy combinations, which we call "simple trigger strategy" combinations. Note that Player 1's optimal exit strategy varies depending on the outside option structure.

**Cooperation phase**: If the history is empty or does not have $D$, play $(C, C)$ and Player 1 uses an optimal exit strategy given that $(C, C)$ is repeated as long as the game continues.
**Punishment phase**: If the history contains $D$, play $(D, D)$ and Player 1 uses an optimal exit strategy given that $(D, D)$ is repeated as long as the game continues.

## 3. DETERMINISTIC OUTSIDE OPTION

When the outside option is deterministic, $c > v$ implies that Player 1's optimal exit strategy in the cooperation phase is not to take the option, and $v > d$ implies that the optimal exit strategy in the punishment phase is to take the option at the first opportunity. Therefore, the play path of the simple trigger strategy combination is the eternal cooperation. Let us find the lower bound of $\delta$ that sustains the eternal cooperation, that is, that makes the simple trigger strategy combination a SPE.

Recall that in the ordinary repeated Prisoner's Dilemma with discounting, the eternal cooperation is sustained by the simple trigger strategy without the exit option if and only if

$$\begin{aligned} \frac{c}{1-\delta} &\geqq g + \frac{\delta d}{1-\delta} \\ \iff \delta &\geqq \frac{g-c}{g-d} =: \underline{\delta}. \end{aligned}$$

In our game, Player 1 does not deviate in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq g + \frac{\delta v}{1-\delta} \tag{1}$$

$$\iff \delta \geqq \frac{g-c}{g-v} =: \delta_1^D(v), \tag{2}$$

and Player 2 does not deviate in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq g \iff \delta \geqq \frac{g-c}{g} =: \delta_2^D.$$

Let $\delta^D(v) = \max\{\delta_1^D(v), \delta_2^D\}$. Then the simple trigger strategy combination is a SPE if and only if $\delta \geqq \delta^D(v)$. Moreover, $v > d$ implies that $\delta_1^D(v) > \underline{\delta}$, and

$d \geqq 0$ implies that $\delta_2^D \leqq \underline{\delta}$. Hence $\delta^D(v) = \delta_1^D(v) > \underline{\delta}$. This means that, for any $\delta \in [\underline{\delta}, \delta^D(v))$, the existence of an outside option, greater than the mutual defection payoff, makes the eternal cooperation impossible, while it was possible if the game were an ordinary repeated Prisoner's Dilemma. It is also easy to see that $\delta^D(v)$ is increasing in $v$, implying that better outside option makes it harder to cooperate. Since $\lim_{v \to c} \delta^D(v) = 1$, the range of $\delta$ that sustains the eternal cooperation shrinks to the empty set, as the outside option approaches to $c$.

PROPOSITION 1. *For any $v \in (d, c)$, the eternal cooperation is sustained as the outcome of a SPE if and only if $\delta \geqq \delta^D(v) > \underline{\delta}$. Hence, for any $\delta \in [\underline{\delta}, \delta^D(v))$, the eternal cooperation cannot be sustained in the outside option model, while it is sustainable in the ordinary repeated Prisoner's Dilemma.*

Alternatively, given $\delta \in [\underline{\delta}, 1)$, we can define the highest outside option level $v^*(\delta)$ which makes Player 1 not to deviate in the cooperation phase by

$$\begin{aligned} \frac{c}{1-\delta} &= g + \delta \frac{v^*}{1-\delta} \\ \Rightarrow \quad v^*(\delta) &:= \frac{1}{\delta}\{c - (1-\delta)g\}. \end{aligned} \tag{3}$$

Clearly, $v^*$ is increasing in $\delta$, $v^*(\underline{\delta}) = d$, and $\lim_{\delta \to 1} v^*(\delta) = c$. Since Player 2 does not deviate for $\delta \geqq \underline{\delta}$, we have the following corollary.

COROLLARY 1. *Given $\delta \geqq \underline{\delta}$, the eternal cooperation cannot be sustained if and only if the outside option $v$ exceeds $v^*(\delta)$.*

Two remarks are in order. First, although we focus on the repeated play of $(C, C)$, one might wonder that if players play $(D, C)$ occasionally, it may reduce the sufficient level of the discount factor. Playing $(D, C)$ has two effects. One is that it is possible to lower the sufficient discount factor for Player 1 to follow the strategy. The other is that Player 2 must have incentive to play $(D, C)$. Therefore it is not always the case that playing $(D, C)$ can reduce the sufficient discount factor. In fact, under some parameter condition, $(C, C)$ is the easiest action profile to sustain. For details see Appendix A.

Second, so far we have assumed that the outside option is a single stationary sequence $\{v, v, \ldots\}$. If different sequences become available over time, the cooperation may fall apart, even if most of outside options are unattractive, i.e., below $v^*(\delta)$.

To see this, fix $\delta$ and suppose that at the end of each period $t$, a sequence $\{v(t), v(t), \ldots\}$ is the outside option and there exists the smallest integer $T < \infty$ such that $v(T) >$

$v^*(\delta)$. That is, $T$ is the first time that the outside option exceeds $v^*(\delta)$. Then at the end of period $T$, Player 1 would exit (given that the players play $(C, C)$ as long as they are in the game), and thus the players would not play $(C, C)$ in $T$.

Therefore, deterministic fluctuations of outside options do not make cooperation easier. By contrast, in the next section we consider stochastic outside options such that in each period the actual outside option is random. Even if the players know that eventually an attractive option arrives, the uncertainty of the timing can make the mutual cooperation possible until the realization. This is a striking difference from the above deterministic and fluctuating option case.

## 4. STOCHASTIC OUTSIDE OPTIONS

In this section we consider the case that Player 1 receives stochastic outside options at the end of each period from an i.i.d. distribution. The randomness can be interpreted several ways, such as subjective uncertainty, external perturbation, or a draw from a distribution of options. To make the comparison with the deterministic case, throughout this section we fix the mean of the distribution equal to $v \in (d, c)$.

The stochasticity of the options changes both the value of cooperation phase and the value of the punishment phase so that in addition to the eternal cooperation and no cooperation, the stochastic cooperation (cooperation until a stochastic end of the game) may become the play path. Then it is possible that the volatility of the payoffs changes the on-path play from no cooperation to the stochastic cooperation, as we discussed in Introduction. Note that, in the single deterministic outside option model, under the myopic equilibrium punishment, $(C, C)$ is repeatedly played for a finite number of periods if and only if the eternal cooperation is sustained. However, in the stochastic outside option model, this equivalence does not hold.

### 4.1. *Symmetric Binary Distributions*

In this subsection we focus on a simple binary distribution of outside options: $\{v + \epsilon, v + \epsilon, \ldots\}$ and $\{v - \epsilon, v - \epsilon, \ldots\}$ for some $\epsilon > 0$, which becomes available with probability $1/2$ each.[8] With this formulation we can see the effect of the mean $v$ and the spread $\epsilon$ of the distribution separately.

[8]Alternatively we can assume that only the next period option becomes known and future option values are still random. If the one-shot payoff of $v + \frac{\epsilon}{1-\delta}$ and $v - \frac{\epsilon}{1-\delta}$ obtain with equal probability in each period, then taking the option $v + \frac{\epsilon}{1-\delta}$ (resp. $v - \frac{\epsilon}{1-\delta}$) in a period and receiving the same random sequence afterwards gives the same expected payoff of $\frac{v+\epsilon}{1-\delta}$ (resp. $\frac{v-\epsilon}{1-\delta}$).

Unlike the deterministic outside option cases analyzed so far, it may not be optimal for Player 1 to exit immediately in the punishment phase under the stochastic options, even though the average is still $v > d$. Thus we first clarify the optimal exit strategy for Player 1 in the cooperation phase and in the punishment phase respectively, and then we derive the lower bound of $\delta$.

Suppose that repeated $(C, C)$ is expected as long as the game continues. Since $c > v - \epsilon$, in any period, either taking only the option of $v + \epsilon$ or not taking any option is the optimal strategy. Let $V$ be the continuation payoff, measured at an exit decision node, when Player 1 takes only the option of $v + \epsilon$. While waiting for the good option the players play $(C, C)$ repeatedly. Thus $V$ satisfies the following recursive equation:

$$V = \frac{1}{2}\Big(\frac{v+\epsilon}{1-\delta}\Big) + \frac{1}{2}(c + \delta V).$$

To explain, with probability 1/2, Player 1 receives the good option $v + \epsilon$ which she takes and thus the continuation payoff becomes $(v + \epsilon)/(1 - \delta)$. With probability 1/2, Player 1 receives the bad option $v - \epsilon$ in which case she stays in the game and follows $(C, C)$ in the next period and faces the same distribution of the outside options at the end of the next period. In this case the continuation payoff is $c + \delta V$. Explicitly, we have

$$V = \frac{\frac{1}{2}(\frac{v+\epsilon}{1-\delta}) + \frac{1}{2}c}{1 - \frac{\delta}{2}} = \frac{\frac{v+\epsilon}{1-\delta} + c}{2 - \delta}. \tag{4}$$

If Player 1 does not take any outside option, the continuation payoff is $c/(1 - \delta)$. Therefore, not taking any option is optimal in the cooperation phase if and only if

$$\frac{c}{1-\delta} \geqq V \iff (2-\delta)c \geqq v + \epsilon + (1-\delta)c \qquad \text{from (4)}$$
$$\iff c \geqq v + \epsilon.$$

In summary we have the following characterization of the optimal exit strategy in the cooperation phase.

Lemma 2. *When $(C, C)$ is expected as long as the game continues, not taking any outside option is the optimal exit strategy for Player 1 if $c \geqq v + \epsilon$, and taking only the good option $v + \epsilon$ is optimal otherwise.*

Analogously, suppose that repeated $(D, D)$ is expected as long as the game continues. Since $d < v + \epsilon$, either taking only the option of $v + \epsilon$ or taking any option is the

| | | $\epsilon \leqq v-d$ | | $v-d<\epsilon$ |
|---|---|---|---|---|
| | | $\delta \leqq \delta^P(v,\epsilon)$ | $\delta^P(v,\epsilon) \leqq \delta$ | |
| $c \geqq v+\epsilon$ | cooperation phase | No exit | No exit | |
| | punishment phase | Take any option | Take only $v+\epsilon$ | |
| $v+\epsilon > c$ | cooperation phase | Take only $v+\epsilon$ | Take only $v+\epsilon$ | |
| | punishment phase | Take any option | Take only $v+\epsilon$ | |

Table 3: Player 1's Optimal Exit Strategy

optimal exit strategy. Let $W$ be the continuation payoff, measured at an exit decision node, when Player 1 takes only the option of $v+\epsilon$. While waiting for the good option the players play $(D,D)$ repeatedly. Thus $W$ satisfies

$$W = \frac{1}{2}\Big(\frac{v+\epsilon}{1-\delta}\Big) + \frac{1}{2}(d+\delta W),$$

and hence

$$W = \frac{\frac{v+\epsilon}{1-\delta}+d}{2-\delta}. \tag{5}$$

If Player 1 exits immediately by taking any outside option, the continuation payoff is $\frac{1}{2}(\frac{v+\epsilon}{1-\delta}) + \frac{1}{2}(\frac{v-\epsilon}{1-\delta}) = \frac{v}{1-\delta}$. Therefore, waiting for the good option $v+\epsilon$ in the punishment phase is optimal if and only if

$$W \geqq \frac{v}{1-\delta} \iff \frac{\frac{v+\epsilon}{1-\delta}+d}{2-\delta} \geqq \frac{v}{1-\delta} \qquad \text{from (5)}$$

$$\iff \delta \geqq \frac{v-d-\epsilon}{v-d}. \tag{6}$$

Let $\delta^P(v,\epsilon) = \max\{\frac{v-d-\epsilon}{v-d}, 0\}$. The superscript P stands for the punishment phase. Thus, we have the following characterization of the optimal exit strategy in the punishment phase. The optimal exit strategies are also summarized in Table 3.

Lemma 3. *When $(D,D)$ is expected as long as the game continues, taking only the good outside option of $v+\epsilon$ is the optimal exit strategy for Player 1 if $\delta \geqq \delta^P(v,\epsilon)$, and taking any outside option is optimal otherwise.*

We now find the lower bound of the discount factor $\delta$ to sustain repeated mutual cooperation *as long as possible*, using the simple trigger strategy combination described in Section 3. Lemma 2 implies that when $c \geqq v+\epsilon$, the eternal cooperation is possible, while when $v+\epsilon > c$, only the stochastic cooperation is possible. Therefore we explain the intuition of the characterization of the lower bound of $\delta$ for two cases separately. The formal proof is in Appendix B.

First, consider the case that $c \geqq v + \epsilon$ so that the optimal value of the cooperation phase is $\frac{c}{1-\delta}$. The value of the optimal one-shot deviation is

$$\max\{g + \delta \frac{v}{1-\delta}, g + \delta W\}.$$

As $\delta$ increases from 0 to 1, both of these values increase, but the value of the cooperation phase is more convex than the optimal one-shot deviation value. (See Figure 2.[9]) There are two cases of how $c/(1-\delta)$ intersects with the deviation value.

If $\delta^D(v) \leqq \delta^P(v, \epsilon)$, that is,

$$\delta^D(v) \leqq \delta^P(v, \epsilon) \iff \epsilon \leqq \frac{(v-d)(c-v)}{g-v}, \tag{7}$$

then $c/(1-\delta)$ intersects with the one-shot deviation value when the latter is $g + \delta v/(1-\delta)$, as shown in Figure 2(a). In this case, the eternal cooperation is sustained if and only if

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W, g + \delta \frac{v}{1-\delta}\} = g + \delta \frac{v}{1-\delta} \iff \delta \geqq \delta^D(v).$$

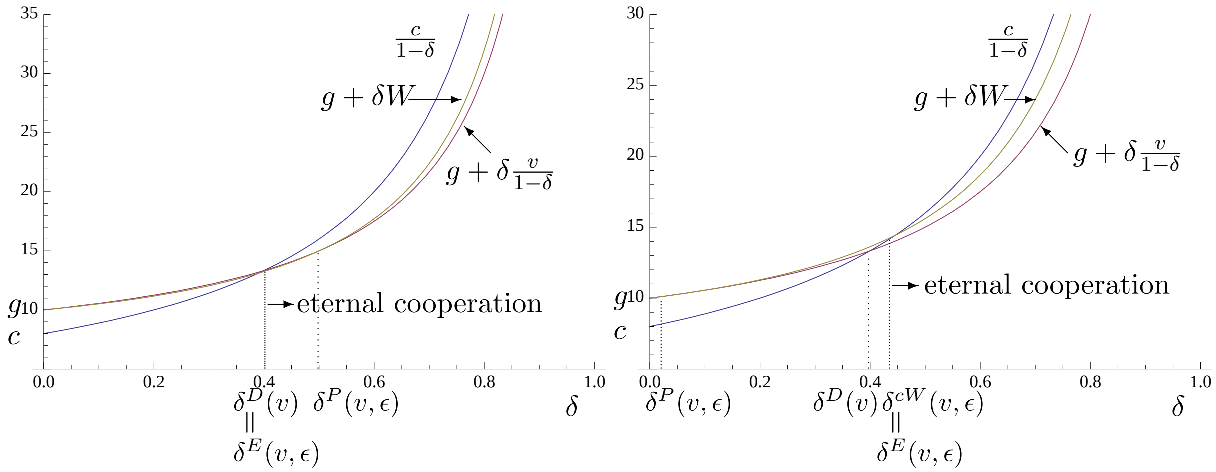
By contrast, if $\epsilon$ is large, so that $\delta^D(v) > \delta^P(v, \epsilon)$ (but still $c \geqq v + \epsilon$), then $c/(1-\delta)$ intersects with the deviation value when the latter is $g + \delta W$, as shown in Figure 2(b). Let $\delta^{cW}(v, \epsilon)$ be the solution to

$$\frac{c}{1-\delta} = g + \delta W.$$

Then this $\delta^{cW}(v, \epsilon)$ is the lower bound of the discount factors to sustain the eternal cooperation in the case of $\delta^D(v) > \delta^P(v, \epsilon)$. Since $g + \delta W > g + \delta \frac{v}{1-\delta}$ in this case, when $\frac{c}{1-\delta}$ intersects with the deviation value, $\delta^{cW}(v, \epsilon)$ is strictly greater than $\delta^D(v)$, as Figure 2(b) shows. Therefore, when $c \geqq v + \epsilon$, the eternal cooperation is sustained[10] if and only if $\delta$ is not less than

$$\delta^E(v, \epsilon) := \max\{\delta^D(v), \delta^{cW}(v, \epsilon)\}$$

and this lower bound is never smaller than $\delta^D(v)$. Note that it is possible to sustain the stochastic cooperation (to cooperate until $v + \epsilon$ realizes) under $c \geqq v + \epsilon$ as well,




2(a): $\epsilon \leqq (v-d)(c-v)/(g-v)$ 2(b): $(v-d)(c-v)/(g-v) < \epsilon$

Figure 2: Less Cooperation under Stochastic Outside Options ($c \geqq v+\epsilon$)

but higher $\delta$ is needed because the value of the stochastic cooperation is smaller than $c/(1-\delta)$.

Second, consider the case that $v+\epsilon > c$ so that the optimal value of the cooperation phase is $c+\delta V$. Note that in this case $c+\delta V$ is uniformly greater than the eternal cooperation value $\frac{c}{1-\delta}$ for any $\delta > 0$. Again, there are two possibilities of how the cooperation value intersects with the deviation value, $\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\}$. If $c+\delta V$ intersects with the deviation value when the latter is $g+\delta\frac{v}{1-\delta}$, let $\delta^V(v,\epsilon)$ be the solution to

$$c+\delta V = g+\delta\frac{v}{1-\delta}.$$

Then $c+\delta V > c/(1-\delta)$ for all $\delta > 0$ implies that $\delta^V(v,\epsilon) < \delta^D(v)$. (See Figure 3(a).[11])

If $c+\delta V$ intersects with the deviation value when the latter is $g+\delta W$, the intersection is computed as follows:

$$c+\delta V = g+\delta W \iff \delta(V-W) = g-c \iff \delta = \frac{2(g-c)}{g-d}. \tag{8}$$

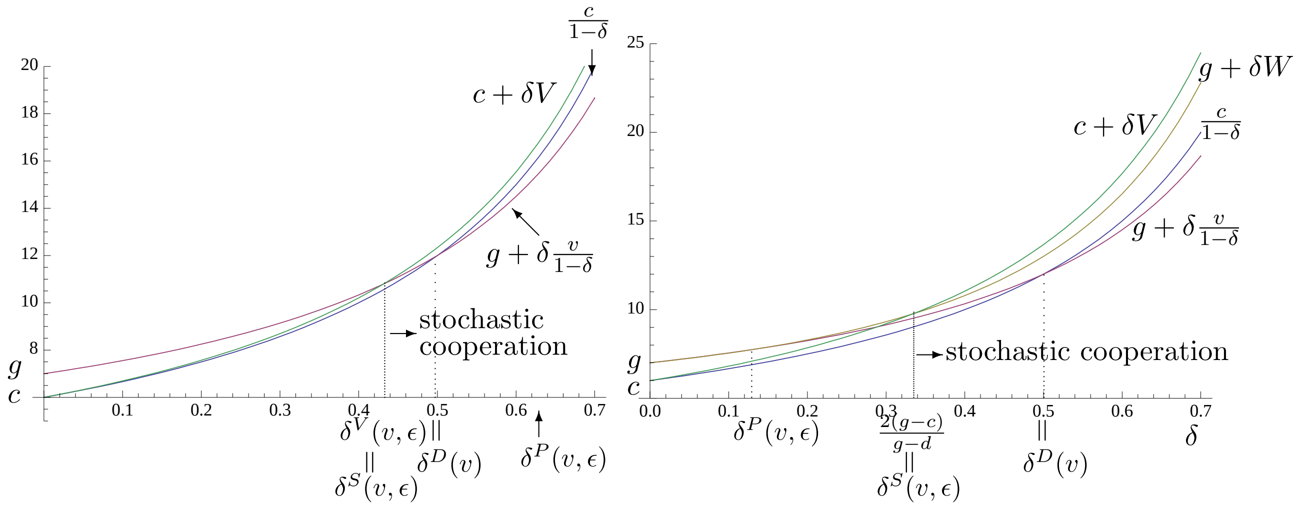
Notice that $v > (g+d)/2$ if and only if $\frac{2(g-c)}{g-d} < \delta^D(v)$. Let us define

$$\delta^S(v,\epsilon) := \max\{\delta^V(v,\epsilon), \frac{2(g-c)}{g-d}\}.$$

[9]The parameter values are $(g,c,d,\ell,v) = (10,8,3,2,5)$ and $\epsilon = 1$ for 2(a) and $\epsilon = 1.9$ for 2(b).

[10]To be precise, Player 2's deviations must be checked. This is done in the formal proof of Proposition 2.

[11]The parameter values are $(g,c,d,\ell,v) = (7,6,1,0.1,5)$ and $\epsilon = 1.5$ for 3(a) and $\epsilon = 3.5$ for 3(b).




3(a): $\delta^P(v,\epsilon)$ large

3(b): $\delta^P(v,\epsilon)$ small

Figure 3: More Cooperation under Stochastic Outside Options ($v+\epsilon > c$)

Then the stochastic cooperation is sustained if and only if $\delta \geqq \delta^S(v,\epsilon)$. Since $\delta^V(v,\epsilon) < \delta^D(v)$, the lower bound $\delta^S(v,\epsilon)$ is strictly smaller than $\delta^D(v)$ if and only if $v > (g+d)/2$.

Therefore, even when one of the options is very attractive, if the one-shot gain $g$ from defection is not too large, then Player 1 with a mid-range $\delta$ who would not cooperate under the deterministic option would cooperate under the stochastic outside options, as long as she is in the game. This is because the value of stochastic cooperation, $c + \delta V$, is increased by the outside option more than that of the optimal deviation.

PROPOSITION 2. *Case 1: Suppose that $c \geqq v+\epsilon$. Then the eternal cooperation is sutsained if and only if $\delta \geqq \delta^E(v,\epsilon)$, where $\delta^E(v,\epsilon) \geqq \delta^D(v)$.*
*Case 2: Suppose that $v+\epsilon > c$. Then the stochastic cooperation is sustained if and only if $\delta \geqq \delta^S(v,\epsilon)$. Moreover, $\delta^S(v,\epsilon) < \delta^D(v)$ if and only if $v > (g+d)/2$.*

Proof: See Appendix B.

Note, however, that even though $\delta^S(v,\epsilon) < \delta^D(v)$ for Case 2, still $\delta^S(v,\epsilon) \geqq \frac{2(g-c)}{g-d} > \frac{g-c}{g-d} = \underline{\delta}$ holds. Therefore, the existence of outside options in any form makes it more difficult to achieve mutual cooperation than in the ordinary repeated game. The "locked-in" feature of repeated games is a strong device to enforce mutual cooperation.

### 4.2. *Mean Effect and Perturbation Effect*

Let us consider comparative statics when the mean $v$ or the spread $\epsilon$ of the outside option distribution changes. Essentially, the increase of the mean $v$ increases the option value of the punishment phase more than that of the cooperation phase. Therefore the increase of the mean makes cooperation more difficult, just like in the deterministic option case. By contrast, the change of the spread $\epsilon$ is relevant only when Player 1 wants to wait for the good option. Therefore the perturbation effect of $\epsilon$ is not apparent and is in fact quite complex, which we show below.

The increase of the mean $v$ of the outside options is always a bad news for cooperation. To see this, let us distinguish two ranges: $v + \epsilon \leqq c$ and $v + \epsilon > c$, given an $\epsilon$. When $v$ is small so that $v + \epsilon \leqq c$ (i.e., the on-path value is $c/(1-\delta)$), only the deviation value $\max\{g + \delta W, g + \delta\frac{v}{1-\delta}\}$ increases as $v$ increases. Hence $\delta^E(v, \epsilon)$ is increasing in $v$.

In the range of $v + \epsilon > c$, the on-path value $c + \delta V$ also increases as $v$ increases, so that we need to see the relative change between the on-path value and the punishment phase. The relationship between $c + \delta V$ and $g + \delta W$ does not change when $v$ changes, since the critical value of $\delta$ is $\frac{2(g-c)}{g-d}$ and is independent of $v$. The relationship between $c + \delta V$ and $g + \delta\frac{v}{1-\delta}$ is seen as follows.

$$\begin{aligned} & c + \delta V \geqq g + \delta\frac{v}{1-\delta} \\ \iff \quad & \delta\frac{(1-\delta)(c-v)+\epsilon}{(2-\delta)(1-\delta)} \geqq g - c. \end{aligned} \tag{9}$$

Notice that the LHS of (9) is increasing in $\delta$ and decreasing in $v$. Therefore, $\delta^V(v, \epsilon)$ is increasing in $v$. This means that although $v$ increases both the value of stochastic cooperation $c+\delta V$ and the deviation value $g+\delta\frac{v}{1-\delta}$, the increase in the latter dominates. In sum, the increase in the mean of the outside options always makes cooperation more difficult. This is consistent with the deterministic case.

By contrast, the perturbation effect of $\epsilon$ is more complex, since it only affects the value when Player 1 wants to wait for the good option, i.e., given $\delta$ and $v$, the increase of $\epsilon$ increases $V$ and $W$ only. There are two important thresholds for $\epsilon$. First, $\epsilon \leqq c - v$ implies that $\frac{c}{1-\delta}$ is the cooperation value, and $\epsilon > c - v$ implies that $c + \delta V$ is the cooperation value. Second, recall that

$$\delta^P(v, \epsilon) < \delta^D(v) \iff \epsilon > \frac{(v-d)(c-v)}{g-v},$$

which changes the optimal deviation value. Depending on whether $v > (g+d)/2$ or not, the two thresholds are ordered differently:

$$v > \frac{g+d}{2} \iff 0 < c - v < \frac{(v-d)(c-v)}{g-v};$$
$$v \leqq \frac{g+d}{2} \iff 0 < \frac{(v-d)(c-v)}{g-v} \leqq c - v.$$

Moreover, as we recall from (8),

$$v > \frac{g+d}{2} \iff \frac{2(g-c)}{g-d} < \delta^D(v).$$

Therefore, we have two fundamentally different cases depending on whether $v > (g+d)/2$ or not.

Case 1: $v > (g+d)/2$, so that $0 < c - v < \frac{(v-d)(c-v)}{g-v}$.

For any $\epsilon \in [0, c-v]$, $\delta^D(v) < \delta^P(v, \epsilon)$ which implies that

$$\max\{g + \delta W, g + \delta \frac{v}{1-\delta}\} = g + \delta \frac{v}{1-\delta}.$$

Hence the lower bound is $\delta^D(v)$.

For any $\epsilon > c-v$, the optimal on-path value is $c+\delta V$, which increases as $\epsilon$ increases. Clearly, as long as $\epsilon \leqq \frac{(v-d)(c-v)}{g-v}$, still $\delta^D(v) \leqq \delta^P(v,\epsilon)$ holds so that the critical $\delta$ is when $c+\delta V$ intersects with $g+\delta\frac{v}{1-\delta}$, as we have seen in Figure 3(a). Since the on-path value $c+\delta V$ is increasing in $\epsilon$ but the deviation value is independent of $\epsilon$, this critical value $\delta^V(v,\epsilon)$ is decreasing in $\epsilon$. (This can be also seen from (9).) Thus the lower bound of the discount factors that sustain the stochastic cooperation *decreases* as the spread $\epsilon$ increases. When $\epsilon$ becomes large enough,[12] the relevant lower bound is determined by $c + \delta V = g + \delta W$, which is a constant, $2(g-c)/(g-d)$.

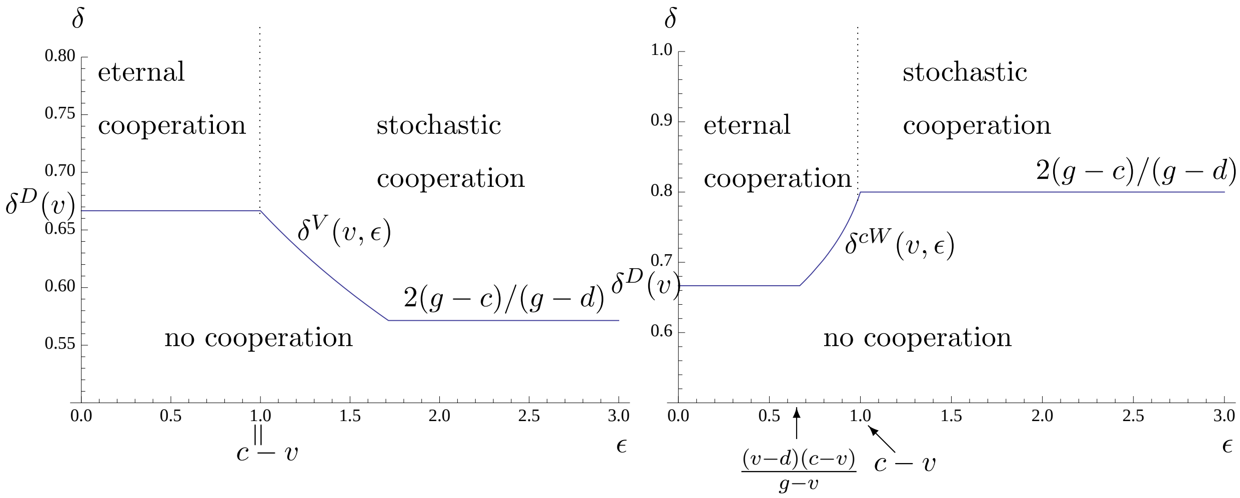
This is graphically shown in Figure 4(a).[13] It shows that for mid-range $\delta \in (\frac{2(g-c)}{g-d}, \delta^D(v))$, the increase of $\epsilon$ changes no cooperation into the stochastic cooperation. Therefore the perturbation enhances cooperation.

Case 2: $v \leqq (g+d)/2$, so that $0 < \frac{(v-d)(c-v)}{g-v} \leqq c - v$.

In this case, we divide $[0, c-v]$ into two intervals, $[0, \frac{(v-d)(c-v)}{g-v}]$ and $(\frac{(v-d)(c-v)}{g-v}, c-v]$. For $\epsilon \in [0, \frac{(v-d)(c-v)}{g-v}]$, as in Case 1, $\delta^D(v)$ is the lower bound. For $\epsilon \in (\frac{(v-d)(c-v)}{g-v}, c-v]$,

[12]The critical level is beyond $\epsilon = \frac{(v-d)(c-v)}{g-v}$, because even if $\delta^P(v,\epsilon) = \delta^D(v)$, that does not imply that $c + \delta V$ intersects with $g + \delta W$ at that $\epsilon$.

[13]The parameter values are $(g, c, d, \ell, v) = (8, 6, 1, 0.9, 5)$




4(a): $v > (g+d)/2$ 4(b): $v \leqq (g+d)/2$

Figure 4: Perturbation Effect

$\delta^P(v,\epsilon) < \delta^D(v)$ holds so that the critical $\delta$ is when $c/(1-\delta)$ intersects with $g+\delta W$, as in Figure 2(b). This $\delta^{cW}(v,\epsilon)$ is increasing in $\epsilon$ since the deviation value $g+\delta W$ is increasing in $\epsilon$, while $c/(1-\delta)$ is independent of $\epsilon$.

When $\epsilon > c-v$, then the optimal on-path value is $c+\delta V$ and the optimal deviation value is still $g+\delta W$. Thus the lower bound is a constant, $2(g-c)/(g-d)$. This is graphically shown in Figure 4(b).[14] It shows that for any $\delta$, the increase in $\epsilon$ does not enhance cooperation.

To highlight the positive effect of perturbation, we summarize as follows.

COROLLARY 2. *When $v > (g+d)/2$, for any $\delta$ such that $\frac{2(g-c)}{g^{d})} < \delta < \delta^D(v)$, there exists $\underline{\epsilon}(\delta)$ (the solution to $\delta^V(v,\epsilon) = \delta$) such that for any $\epsilon \geqq \underline{\epsilon}(\delta)$, stochastic cooperation is sustained while for any $0 \leqq \epsilon < \underline{\epsilon}(\delta)$ no cooperation is possible.*

The ordinary repeated game literature looks only at the vertical axis of Figure 4, where $\epsilon = 0$, and the case of $v = d$. By adding the dimension of $(v,\epsilon)$, we enlarged the scope of the analysis and found the positive effect of payoff perturbation.

Our result is different from the effect of stochastic discount factor (Dal Bó, 2007), which affects both the cooperation phase value and the punishment phase value, and that of stochastic payoffs in Rotemberg and Saloner (1986). As we discussed in Introduction, their results can be interpreted as the eternal cooperation being more difficult under volatility. We have provided a third source of volatility via the outside options

[14] The parameter values are $(g,c,d,\ell,v) = (8,6,3,0.9,5)$

and expanded the notion of "repeated cooperation" to include not only the eternal cooperation but also the stochastic cooperation. Then we can show that in some cases cooperation is enhanced under more volatility.

Yasuda and Fujiwara-Greve (2009) shows a similar result for ordinary repeated games with perturbed payoffs. Essentially, if the volatility of the payoffs takes the form that stopping cooperation in that period is beneficial, then players can still *selectively cooperate* in some periods, even if they cannot cooperate under no perturbation.

### 4.3. *General Binary Distributions with a Preserved Mean*

We extend the analysis to a general binary distribution to incorporate more realistic situations, as well as to generalize the arrival probability of the attractive option. For example, in employment relationships, most of the time the outside option is not so good, but once in a while a very attractive outside option may arrive. If the probability of the good outside option is very small, there is not much discrepancy between the stochastic cooperation and the eternal cooperation, and the change in the lower bound of the discount factor has a significant meaning.

Although there are many ways to formulate a general binary distribution with a fixed mean $v$, we use the following formulation. Suppose that there are two outside options $v^+ > v^-$, which obtain with probability $p$ and $1-p$ respectively at the end of each period. As before, the option $v^+$ (resp. $v^-$) indicates that a stationary payoff sequence $\{v^+, v^+, \ldots\}$ (resp. $\{v^-, v^-, \ldots\}$) is given, or a one-shot payoff of $\frac{v^+}{1-\delta}$ (resp. $\frac{v^-}{1-\delta}$) is given. To keep the mean $v = pv^+ + (1-p)v^-$ between $d$ and $c$, we fix[15] $v^-$ $(< v)$ and $v \in (d, c)$ and let $v^+(p) = (v - v^-)/p + v^-$. Note that $v^+(p)$ becomes a decreasing function of $p > 0$. For notational simplicity we often write $v^+$ when there is no danger of confusion. As before we find conditions for the simple trigger strategy combination to be a SPE.

Let $V(p)$ (given $v$ and $v^-$) be the value in the cooperation phase, measured at the end of a period before a stochastic option arrives, when Player 1 takes only the good outside option $v^+$ in any period during the cooperation phase. It has the following recursive structure.

$$V(p) = p\frac{v^+}{1-\delta} + (1-p)\{c + \delta V(p)\}.$$

[15]If we fix $v^+$ instead, then the decrease of lower bound as $p$ increases is rather obvious, since $p$ increases the value of waiting for the better option in both cooperation phase and punishment phase in the same way. A more interesting case is the one we analyze here, in which the good option's value decreases when its probability increases.

Therefore

$$V(p) = \frac{pv^+ + (1-p)(1-\delta)c}{(1-\delta)\{1-(1-p)\delta\}}. \tag{10}$$

Using this, we characterize the optimal exit strategy in the cooperation phase. The proof is essentially the same as that of Lemma 2 and is thus omitted.

LEMMA 4. *When $(C, C)$ is expected as long as the game continues, not taking any outside option is the optimal exit strategy for Player 1 if $c \geqq v^+$, and taking only $v^+$ (and therefore exiting with probability $p$) is optimal otherwise.*

Therefore, the only condition to determine the optimal exit strategy in the cooperation phase is whether the best option exceeds $c$, regardless of its probability. This is a generalization of Lemma 2. In fact, this can be generalized for continuous distributions as well. See Section 5.2.

If $(D, D)$ is expected forever after, the optimal exit strategy for Player 1 depends on the outside option distribution as follows. Let $W(p)$ be the value when Player 1 takes only the better outside option $v^+$ during the punishment phase (given $v$ and $v^-$). It satisfies

$$W(p) = p\frac{v^+}{1-\delta} + (1-p)\{d + \delta W(p)\}.$$

Hence

$$W(p) = \frac{pv^+ + (1-p)(1-\delta)d}{(1-\delta)\{1-(1-p)\delta\}}. \tag{11}$$

By the same logic of Lemma 3, the optimal exit strategy in the punishment phase is to wait for the good option $v^+$ if and only if

$$W(p) \geqq \frac{v}{1-\delta} \iff \delta v + (1-\delta)d \geqq v^-.$$

Thus we have the following characterization.

LEMMA 5. *When $(D, D)$ is expected as long as the game continues, waiting for the better outside option $v^+$ is the optimal exit strategy for Player 1 if*

$$\delta v + (1-\delta)d \geqq v^-, \quad \text{or} \quad \delta \geqq \max\{\frac{v^- - d}{v - d}, 0\}, \tag{12}$$

*and taking any outside option is optimal otherwise.*

Note that (12) does not depend on the probability of the good option $p$ and is a generalization of (6). As before, we divide the analysis into two cases: $v^+ \leqq c$ so that the eternal cooperation is possible and $v^+ > c$ so that only the stochastic cooperation is possible.

First, suppose that $v^+ \leqq c$, i.e., $p \geqq (v - v^-)/(c - v^-)$. The optimal value function on the play path is $c/(1-\delta)$. The value of a one-step deviation is $g + \delta W(p)$ if (12) holds, and it is $g + \delta v/(1-\delta)$ otherwise. Let $\delta^{cW}(p)$ be the critical discount factor that satisfy

$$\frac{c}{1-\delta} = g + \delta W(p).$$

Then the eternal cooperation is sustained, i.e.,

$$\frac{c}{1-\delta} \geqq \max\{g + \delta W(p), g + \delta \frac{v}{1-\delta}\} \tag{13}$$

is satisfied for any $\delta \geqq \max\{\delta^{cW}(p), \delta^D(v)\} =: \delta^E(p)$, which is not smaller than $\delta^D(v)$. This is a generalization of Proposition 2, Case 1.

Second, suppose that $v^+ > c$, or $p < (v - v^-)/(c - v^-)$. We investigate the lower bound of $\delta$ that sustains the stochastic cooperation, i.e., that satisfy

$$c + \delta V(p) \geqq \max\{g + \delta W(p), g + \delta \frac{v}{1-\delta}\}. \tag{14}$$

For the range of $\delta$ such that $v/(1-\delta) \geqq W(p)$, the increase in the on-path value implies that the lower bound of $\delta$ is less than $\delta^D(v)$. Formally, let $\delta^V(p)$ be the solution to

$$c + \delta V(p) = g + \delta \frac{v}{1-\delta}.$$

Then $\delta^V(p) < \delta^D(v)$ since $c + \delta V(p) > \frac{c}{1-\delta}$ for any $\delta > 0$. For the range of $\delta$ such that $W(p) > v/(1-\delta)$, both on-path value and the punishment value increase, as compared to the deterministic case. Let us find the smallest $\delta$ that satisfy

$$c + \delta V(p) \geqq g + \delta W(p).$$

By computation,

$$\begin{aligned}
& c + \delta V(p) \geqq g + \delta W(p), \\
\iff & \delta\{V(p) - W(p)\} \geqq g - c, \\
\iff & \delta \frac{\{pv^+ + (1-p)(1-\delta)c\} - \{pv^+ + (1-p)(1-\delta)d\}}{(1-\delta)\{1-(1-p)\delta\}} \geqq g - c, \\
\iff & \delta(1-p)(c-d) \geqq \{1-(1-p)\delta\}(g-c), \\
\iff & \delta \geqq \frac{g-c}{(1-p)(g-d)} =: \delta^{VW}(p).
\end{aligned} \tag{15}$$

Therefore

$$\frac{g-c}{(1-p)(g-d)} < \delta^D(v) \iff v > pg + (1-p)d. \tag{16}$$

The condition (16) is a generalization of $v > (g+d)/2$ in Proposition 2. Since $g > c > v > d$, it is easier to be satisfied for small $p$, namely when $p < (v-d)/(g-d)$. (Note that this condition is compatible with $v^+(p) > c$.) In summary we have the following generalization of Proposition 2.

PROPOSITION 3. *Case 1: Suppose that $c \geqq v^+(p)$. Then the eternal cooperation is sustained if and only if $\delta \geqq \delta^E(p)$ and $\delta^E(p) \geqq \delta^D(v)$.*
*Case 2: Suppose that $v^+(p) > c$. Then the stochastic cooperation with exit probability $p$ is sustained if and only if $\delta \geqq \delta^S(p) := \max\{\delta^{VW}(p), \delta^V(p)\}$. Moreover, $\delta^S(p) < \delta^D(v)$ if and only if $v > pg + (1-p)d$.*

Proof: See Appendix B.

Let us investigate the effect of $p$. Since $p$ changes both the probability of the good option $v^+(p)$ as well as its value, the effect of $p$ is clearly not monotonic. We are most interested in the case when $p$ is very small so that the stochastic cooperation is almost the eternal cooperation. Recall that when $v^+(p) > c$, the stochastic cooperation (that ends with probability $p$) is sustained if and only if

$$\delta \geqq \max\{\delta^{VW}(p), \delta^V(p)\} =: \delta^S(p).$$

From (15), $\delta^{VW}(p)$ is increasing in $p$. To check that $\delta^V(p)$ is also increasing, note that by computation, $V(p)$ is decreasing in $p$ for any $\delta$:

$$\frac{\partial V(p)}{\partial p} = \frac{1}{(1-\delta)\{1-\delta+p\delta\}^2}\Big[v^- - \delta v - (1-\delta)c\Big] < 0.$$

Since $g + \delta\frac{v}{1-\delta}$ does not change as $p$ changes, the intersection with $c + \delta V(p)$ moves to the right, i.e., $\delta^V(p)$ is increasing in $p$. Therefore, $\delta^S(p) = \max\{\delta^{VW}(p), \delta^V(p)\}$ is increasing in $p$, when $v^+(p) > c$. Moreover, if $\max\{\frac{v^- - d}{v-d}, 0\} \leqq \underline{\delta}$, then $\delta^S(p) = \delta^{VW}(p)$, and (15) implies that this bound converges to $\underline{\delta}$ as $p$ approaches to 0. Hence we have the following comparative statics result.

PROPOSITION 4. *(i) In the region of $p$ such that $v^+(p) > c$, $\delta^S(p)$ decreases as $p$ decreases.*
*(ii) If $\max\{\frac{v^- - d}{v-d}, 0\} \leqq \underline{\delta}$, then $\lim_{p\to 0} \delta^S(p) = \underline{\delta}$.*

To interpret (i), when the exit probability $p$ becomes very small and $v^+(p)$ becomes very large, cooperation is enhanced in two ways: the duration of the stochastic cooperation becomes longer and the lower bound of the discount factors becomes smaller. The result (ii) means that when $p$ is very small, the outside option has almost no effect. We are back to the repeated game situation because, under the assumption, in both cooperation phase and punishment phase Player 1 waits for the good option but it hardly arrives. This can be also interpreted as robustness of folk theorem in the sense that even if players are free to exit strategically, if the option is negligible the players are *as if* confined in the repeated game and the cooperation is sustained under the same condition.

## 5. EXTENSIONS

### 5.1. *Two-Sided Outside Options*

We extend the model so that Player 2 also has non-negligible outside options. When both players can choose to take outside options, the rule of termination of a repeated game becomes relevant. The unilateral ending rule assumed in the one-sided option model (Table 4(a)) has a specific meaning in the two-sided option model that the repeated game ends if and only if *at least* one player chooses to exit (Table 4(b)). There is an intermediate case of two-sided option model in which both players must agree to end the game, but in that case it is straightforward to prove that any equilibrium outcome of ordinary repeated game can be sustained.[16] Therefore the essentially different models from ordinary repeated games are the one-sided option model and two-sided option model with the unilateral ending rule. Moreover, the unilateral ending rule is the most commonly analyzed rule (e.g., Gosh and Ray, 1996, Kranton, 1996a,b, Fujiwara-Greve, 2002, and Fujiwara-Greve and Okuno-Fujiwara, 2009) and describes well situations such as joint ventures and lender-borrower relationships.

First, consider the deterministic option model. Let $v_1, v_2 \in (d, c)$ be the outside options for Player 1 and Player 2 respectively. By the same argument as in Section 3,

[16]For example, repeated $(C, C)$ can be achieved by the following strategy combination if two players must agree to end the game: Play $C$ and do not take outside options as long as no one played $D$. If someone played $D$ in the past, play $D$ and do not take outside options. Since one player cannot unilaterally end the game to escape, the strategy combination is a subgame perfect equilibrium if and only if the usual grim-trigger strategy combination is a subgame perfect equilibrium in the ordinary repeated Prisoner's Dilemma.

| P1 \ P2 | |
|---|---|
| Stay | Continue |
| Exit | End |

4(a): One-sided Option for P1

| P1 \ P2 | Stay | Exit |
|---|---|---|
| Stay | Continue | End |
| Exit | End | End |

4(b): Two-sided Option with Unilateral Ending Rule

Table 4: Game Continuation Patterns

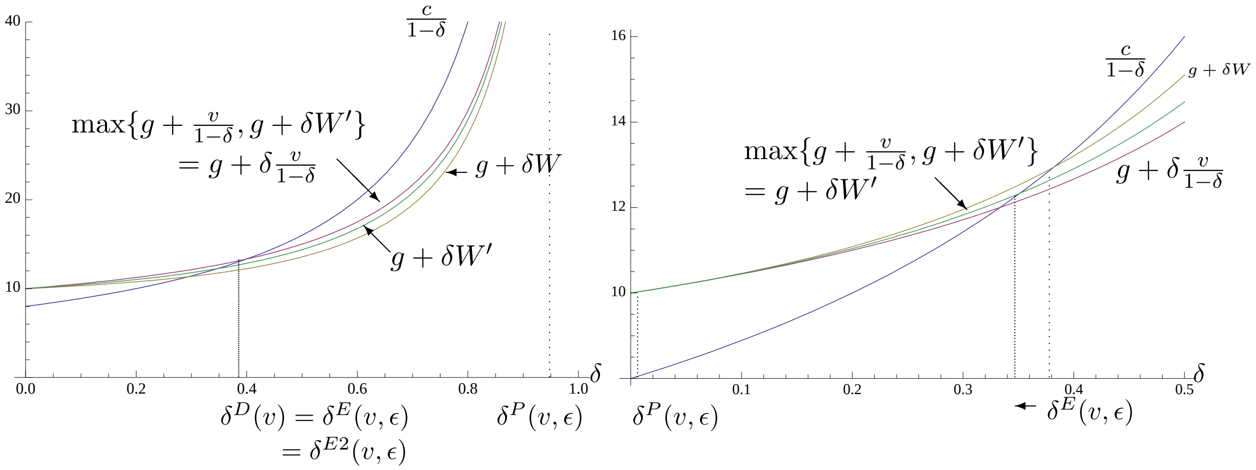
the maximal equilibrium punishment is to exit immediately after the observation of $D$. Under this punishment, Player $i$ would not play $C$ if $\delta < \delta_i^D(v_i) =: \frac{g-c}{g-v_i}$. The range of discount factors that sustains mutual cooperation is $\delta \geqq \max\{\delta_1^D(v_1), \delta_2^D(v_2)\}$, which is weakly narrower than the one in the one-sided outside option model, since $\delta_i^D(\cdot)$ is increasing. Therefore it becomes more difficult to sustain cooperation when both players have deterministic outside options, since both players must be patient enough to stay and cooperate.

Second, let us consider the case that both Player 1 and Player 2 have fluctuating but deterministic outside options. By the same argument as in Section 3, mutual cooperation falls apart if there is a known time period at which one of the players receives an outside option greater than $v^*(\delta)$. Hence we can interpret that cooperation becomes more difficult in the sense that there are more cases of fluctuating outside options that includes $v > v^*(\delta)$ for at least one player.

Third, suppose that Player 1 and Player 2 independently draw stochastic outside options from the same i.i.d. distribution. Since the qualitative results are the same, we focus on the simple distribution such that $v + \epsilon$ obtains with probability 1/2 and $v - \epsilon$ obtains with probability 1/2, independently to each player. Under the independent draws, a player may take an outside option when the other player does not want to, so that the game ends with a different probability and the payoff becomes different from the one in the one-sided outside option case. Specifically, if both players want to take only $v + \epsilon$ in the punishment phase, the continuation value $W'$, measured at the end of a period, satisfies the following recursive structure.

$$W' = \frac{1}{2}\left(\frac{v+\epsilon}{1-\delta}\right) + \frac{1}{4}\left(\frac{v-\epsilon}{1-\delta}\right) + \frac{1}{4}(d + \delta W'). \tag{17}$$

This is because with probability 1/4, one's option turns out to be $v-\epsilon$ but the partner's turned out to be $v + \epsilon$, in which case the game ends and one ends up with the low




5(a): $\epsilon \leqq (v-d)(c-v)/(g-v)$ 5(b): $(v-d)(c-v)/(g-v) < \epsilon$

Figure 5: More Cooperation under Two-Sided Outside Options ($c \geqq v+\epsilon$)

option.

Lemma 6. *For any* $(v,\epsilon)$*, the one-shot deviation values are ordered as follows.*

$$\delta \leqq \delta^P(v,\epsilon) \;\Rightarrow\; g+\delta\frac{v}{1-\delta} \geqq g+\delta W' \geqq g+\delta W;$$
$$\delta^P(v,\epsilon) \leqq \delta \;\Rightarrow\; g+\delta W \geqq g+\delta W' \geqq g+\delta\frac{v}{1-\delta}.$$

Proof: See Appendix B. (See also Figure 5.[17])

Since the punishment phase value is now $\max\{g+\delta W', g+\delta\frac{v}{1-\delta}\}$, it is smaller than the punishment phase value for the one-sided option model. Therefore, when $c \geqq v+\epsilon$ so that the eternal cooperation is to be sustained, the decrease of the punishment phase value makes the eternal cooperation easier, as Figure 5 shows.

By contrast, when $v+\epsilon > c$, both the value in the cooperation phase and the value in the punishment phase may decrease. The continuation value in the cooperation phase, measured at the end of a period, satisfies

$$V' = \frac{1}{2}\left(\frac{v+\epsilon}{1-\delta}\right) + \frac{1}{4}\left(\frac{v-\epsilon}{1-\delta}\right) + \frac{1}{4}(c+\delta V'), \tag{18}$$

[17] The parameter combination is $(g,c,d,\ell,v,\epsilon) = (10, 8, 0.5, 0.1, 5, 0.1)$ for 5(a) and $(g,c,d,\ell,v,\epsilon) = (10, 8, 0.5, 0.1, 4, 3.4)$ for 5(b).

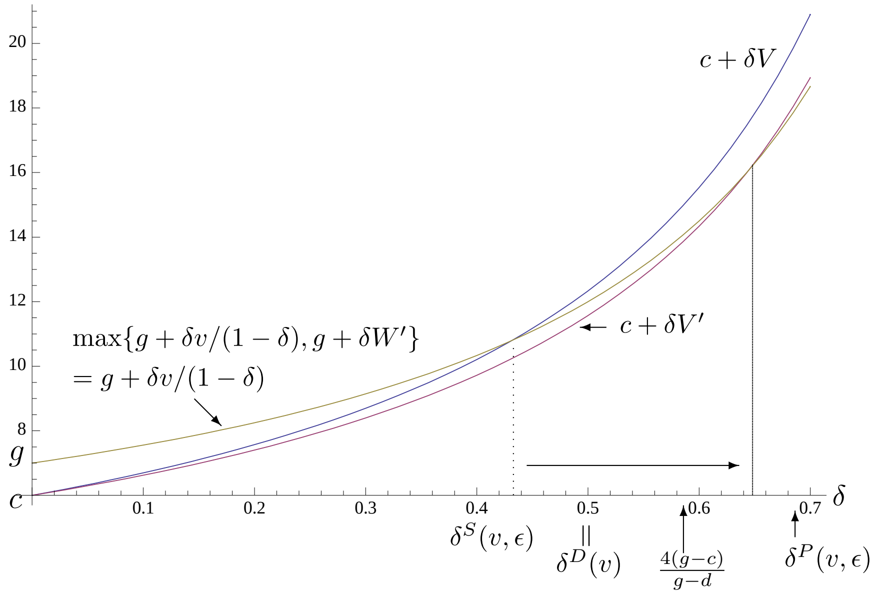



Figure 6: Less Cooperation under Two-Sided Outside Options ($v + \epsilon > c$)

while the one-sided case value can be decomposed as

$$V = \frac{1}{2}\left(\frac{v+\epsilon}{1-\delta}\right) + \frac{1}{4}(c + \delta V) + \frac{1}{4}(c + \delta V).$$

Notice that $c > v$ implies $c + \delta V > (v - \epsilon)/(1 - \delta)$, which in turn implies that $V > V'$ from the above comparison.

Thus, when the stochastic cooperation is to be sustained, both the on-path value $c + \delta V'$ and the punishment phase value, $\max\{g + \delta W', g + \delta\frac{v}{1-\delta}\}$, are reduced, as compared to the one-sided option model. Let $\delta^{V'}(v, \epsilon)$ be the solution to

$$c + \delta V' = g + \delta \frac{v}{1-\delta}.$$

Then $c + \delta V > c + \delta V'$ implies that $\delta^{V}(v, \epsilon) < \delta^{V'}(v, \epsilon)$. On the other hand,

$$c + \delta V' \geqq g + \delta W' \iff \delta(V' - W') \geqq g - c \iff \delta \geqq \frac{4(g-c)}{g-d} > \frac{2(g-c)}{g-d}.$$

Therefore, the lower bound $\delta^{S2}(v, \epsilon) := \max\{\delta^{V'}(v, \epsilon), \frac{4(g-c)}{g-d}\}$ that sustains the stochastic cooperation under the two-sided options is greater than $\delta^{S}(v, \epsilon)$. (See Figure 6.[18])

[18]The parameter combination is $(g, c, d, \ell, v, \epsilon) = (7, 6, 0.2, 0.1, 5, 1.5)$.

Proposition 5. *Case 1: Suppose that $c \geqq v+\epsilon$. Let $\delta^{E2}(v,\epsilon)$ be the lower bound of the discount factors that sustain the eternal cooperation under two-sided outside options. Then $\delta^{E2}(v,\epsilon) \leqq \delta^{E}(v,\epsilon)$.*
*Case 2: Suppose that $v+\epsilon > c$. Let $\delta^{S2}(v,\epsilon)$ be the lower bound of the discount factors that sustain the stochastic cooperation under two-sided outside options. Then $\delta^{S2}(v,\epsilon) > \delta^{S}(v,\epsilon)$.*

For the two-sided option model, we only need to check Player 1's optimization, which is analogous to the one in Proposition 2 and is explained above. Therefore the proof is omitted.

In summary, under two-sided independent stochastic outside options, the effects of perturbation are weakened relative to the one-sided case, because a player may not be able to wait for a good option when she wanted to, which reduces the value of options. However, the weaker effect of perturbation means that the eternal cooperation becomes less difficult and the stochastic cooperation becomes more difficult than the one-sided option case. The weaker effect is obtained as long as the option value is reduced, even if the outside options are not independent.

## 5.2. *Continuum of Outside Options*

The binary distribution models illustrate well the essence of the effect of stochastic outside options on the cooperation within the repeated game. However, it is of some theoretical interest how the model and results extend to a case with a continuum of outside options, which is more standard in some economic models such as search models. We show that the stochastic cooperation is sustained under lower discount factors than those of the deterministic model even under a continuum of outside options.

Let us go back to the one-sided outside option model and assume that Player 1 has a continuum of outside options with the support $[\underline{v}, \overline{v}]$. That is, at the end of each period, an option $x \in [\underline{v}, \overline{v}]$ realizes for Player 1 and if she takes this option, she receives the payoff $x$ forever after, or a one-shot payoff of $\frac{x}{1-\delta}$. Let $F$ be the (differentiable) cumulative distribution function of the outside options and $f$ be its density function. Assume, as before, that the mean outside option $v := \int_{\underline{v}}^{\overline{v}} x f(x) dx$ is strictly between $d$ and $c$.

If Player 1 takes an option of value $x$, then she would also take any option greater than $x$. Hence the optimal exit strategy is a *reservation strategy*: Player 1 takes any outside option not less than a certain level $r$, where $r$ is called the reservation level. Suppose that as long as Player 1 is in the game, she can receive $u$ from the Prisoner's Dilemma, where $u$ can be either $c$ or $d$. Let $U(u,r)$ be the value, at the end of a period before a stochastic outside option realizes, and when Player 1 takes any option not less than $r \in [\underline{v}, \overline{v}]$. It satisfies the following recursive equation:

$$U(u,r) = \int_r^{\overline{v}} \frac{x}{1-\delta} f(x)dx + F(r)\{u + \delta U(u,r)\}. \tag{19}$$

By differentiation of (19) with respect to $r$, we have

$$\begin{aligned} & \frac{\partial U(u,r)}{\partial r} = -\frac{r}{1-\delta} f(r) + f(r)\{u + \delta U(u,r)\} + \delta F(r) \frac{\partial U(u,r)}{\partial r}, \\ \Longleftrightarrow \quad & \frac{\partial U(u,r)}{\partial r} = \frac{f(r)}{1-\delta F(r)} \big[u - \frac{r}{1-\delta} + \delta U(u,r)\big]. \end{aligned}$$

The optimal reservation level, denoted as $r^*(u,\delta)$, is the solution to $\frac{\partial U(u,r)}{\partial r} = 0$ (since the second order condition holds), that is,

$$\frac{r^*(u,\delta)}{1-\delta} = u + \delta U(u, r^*(u,\delta)). \tag{20}$$

This means that the optimal reservation level of the outside options is exactly where Player 1 is indifferent between taking it and not taking it. (19) and (20) imply that

$$\begin{aligned} & \frac{r^*(u,\delta)}{1-\delta} = u + \delta U(u, r^*(u,\delta)) \\ \Longleftrightarrow \quad & r^*(u,\delta) = (1-\delta)u + \delta(1-\delta)\Big\{\int_{r^*(u,\delta)}^{\overline{v}} \frac{x}{1-\delta} f(x)dx + F(r^*(u,\delta))\frac{r^*(u,\delta)}{1-\delta}\Big\}. \end{aligned}$$

Hence, for any $\delta \in (0,1)$ and any $u = c, d$, the optimal reservation level $r^*(u,\delta)$ is the solution to the following equation:

$$r = (1-\delta)u + \delta \int_r^{\overline{v}} x f(x)dx + \delta F(r) r. \tag{21}$$

By differentiation it is straightforward to show that the RHS of (21) is a monotone increasing function of $r$, taking value from $(1-\delta)u + \delta v$ to $(1-\delta)u + \delta\overline{v}$. Figure 7 illustrates this property. Therefore, in the cooperation phase where $u = c$, the optimal reservation level $r^*(c,\delta)$ is less than $\overline{v}$ if and only if $(1-\delta)c + \delta\overline{v} > \overline{v}$, which is equivalent to $\overline{v} > c$.

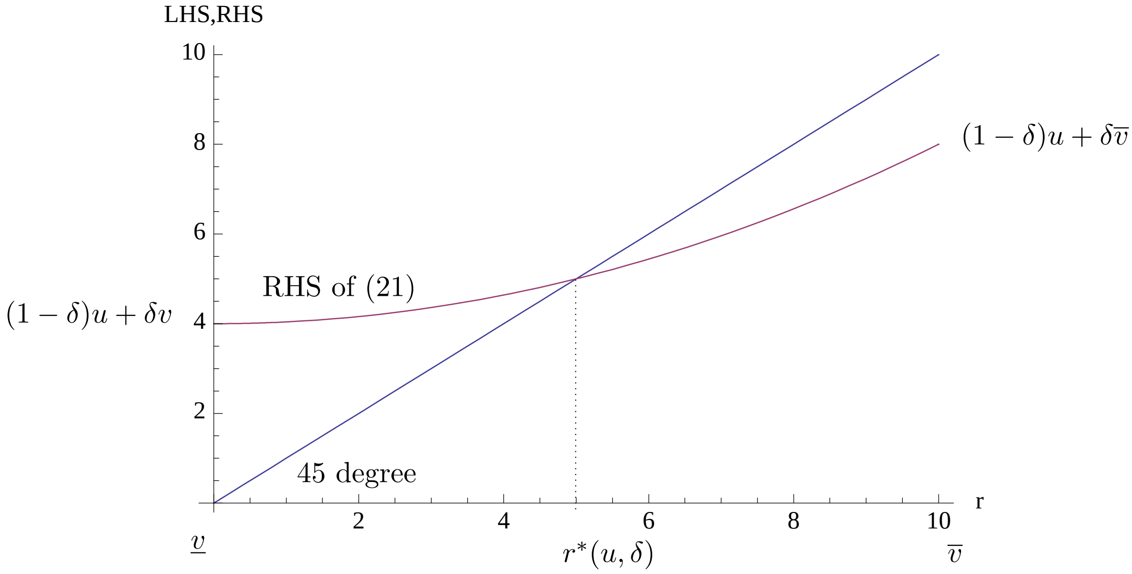



Figure 7: Optimal Reservation Level ($F$ =UNIF$[0, 10]$, $\delta = 0.8$, $u = 0$)

LEMMA 7. *When $(C, C)$ is expected as long as the game continues, the optimal exit strategy for Player 1 is to not to take any outside option if $c \geqq \overline{v}$, and to take any outside option not less than $r^*(c, \delta)$ otherwise.*

In the following we focus on the stochastic cooperation, i.e., we assume that $\overline{v} > c$ and give a sufficient condition under which the lower bound of the discount factors that sustain the stochastic cooperation is less than $\delta^D(v)$.

The equation (21) implies that in the punishment phase when $u = d$, the optimal reservation level is $\underline{v}$ (that is, it is optimal to exit by taking any option) if and only if

$$\underline{v} \geqq (1-\delta)d + \delta \int_{\underline{v}}^{\overline{v}} x f(x)dx + \delta F(\underline{v})\underline{v} = (1-\delta)d + \delta v,$$

which is equivalent to

$$\delta \leqq \frac{\underline{v} - d}{v - d}.$$

This corresponds to $\delta \leqq \delta^P(v, \epsilon)$ for the binary case. If

$$\delta^D(v) \leqq \frac{\underline{v} - d}{v - d} \iff (\underline{v} - d)(g - v) \geqq (g - c)(v - d), \tag{22}$$

then the on-path value $c + \delta U(c, r^*(c, \delta))$ (which is strictly greater than $\frac{c}{1-\delta}$ under the assumption $\overline{v} > c$) intersects with the one-shot deviation value when this is $g + \delta \frac{v}{1-\delta}$, as in Figure 3 (a). Hence the lower bound of the discount factors that deter Player 1's deviation is strictly less than $\delta^D(v)$. In addition, if we impose an extra condition, Player 2 does not deviate either.

Proposition 6. *Assume that $\overline{v} > c$, (22), and $v > \{1 - F(c)\}g$. Let $\delta^F$ be the lower bound of $\delta$ that sustains the stochastic cooperation under the continuum of outside options. Then $\delta^F < \delta^D(v)$.*

Proof: See Appendix B.

We have shown that there is a case of continuum outside options in which the stochastic cooperation is sustained under lower discount factors than those of the deterministic model.

## 6. CONCLUDING REMARKS

Our result can be summarized in three points. First, payoff perturbation may enhance cooperation, which is a new insight. In the literature of ordinary repeated games, only infinitely-repeated cooperation has been analyzed and thus payoff perturbation has negative effect, since perturbation increases the temptation to deviate at some point. However if we extend the notion of "repeated cooperation" to include stochastic repetition of cooperation and the perturbation of outside options is considered, a player wants to wait for a high value, which makes him more patient.

Second, in the simple binary outside option model, the effect of the mean and the spread are quite different. The effect of the mean is monotone and negative in the sense that the lower bound of the discount factors is increasing in $v$. By contrast, the effect of the spread $\epsilon$ is more complex, as shown in Figure 4. For mid-range discount factors and when the deviation gain is not too large, the increase of $\epsilon$ enhances cooperation, while for other parameter combinations, the increase of $\epsilon$ makes cooperation more difficult. Therefore, the option *structure* is important.

Third, in the general binary outside option model, cooperation is enhanced in two ways when a very good option arrives with a very small probability. The small probability implies that the stochastic cooperation is almost the eternal cooperation, and the high value of the good option implies that the lower bound of the discount factors is smaller than the one in the deterministic case. For some parameters, as the probability of the good option converges to 0, the lower bound of the discount factors converges to the one for the ordinary repeated game. Thus our model naturally connects to the repeated Prisoner's Dilemma.

We also found that one-sided and two-sided outside options have different effects. If both players have stochastic outside options, the relative difficulty of cooperation is *weakened* as compared to the one-sided option case. The reason is as follows. If both players can end the game unilaterally, the game ends more frequently and the option value is reduced, since the partner may end the game when one does not want to. This makes the cooperation easier if the punishment phase payoff is reduced but more difficult if the cooperation phase payoff is reduced.

Although the main concern in the present paper is to analyze the sustainability of mutual cooperation under perturbations, it should also be of interest to characterize the set of equilibrium payoffs. Especially, comparative static of the equilibrium payoff sets with respect to the mean value and/or the spread of the outside options has great importance. As we showed in Section 4.2, increased volatility of $\epsilon$ can make Player 1's cooperation easier, which implies that the set of equilibrium payoffs need not be monotonically decreasing (in the sense of set inclusion) in the value of outside options. This non-monotonicity of equilibrium payoffs as the outside options change may have significant implications to applications, for example in policy effects.[19]

Finally, we would like to point out that there is a wide scope of important applications from our analysis. An important implication from our result is that specifications of what players may receive outside of the game, such as potential wage offers or reservation utilities for workers, can have significant effects on their in-game strategic incentives. This finding stands in sharp contrast to the traditional modeling approach in dynamic games and contracting where the outside structure of a game is often assumed to be fixed. We believe that our simple model can provide meaningful insights and implications for many applications.

[19] There is a different non-monotonicity result. In a class of games called exhaustible resource games, Dutta (1995) showed that the first-best outcome is sustainable under a mid-range discount factor but not under high discount factors.

## APPENDIX A: $(C, C)$ CAN BE THE EASIEST TO SUSTAIN

In this Appendix, we give a sufficient condition under which repetition of $(C, C)$ is the easiest stationary action profile to sustain.

For a set $X$, let $\Delta(X)$ be the set of all probability distributions over $X$. The set of feasible payoff combinations is

$$F := \{(u_1, u_2) \in \Re^2 \mid \exists \sigma \in \Delta(\{C, D\} \times \{C, D\}) \text{ such that } u_i = Eu_i(\sigma)\ \forall i = 1, 2\}.$$

For any feasible payoff combination $\mathbf{u} = (u_1, u_2)$, let $\underline{\delta}(\mathbf{u})$ be the lower bound of $\delta$ such that there exists a correlated action profile $\sigma \in \Delta(\{C, D\} \times \{C, D\})$ such that $\mathbf{u} = (Eu_1(\sigma), Eu_2(\sigma))$ and the following strategy combination is a SPE:

**Play Path**: If the history is empty or there was no deviation from $\sigma$ in the past, play $\sigma$ and Player 1 uses an optimal exit strategy given that $\sigma$ is repeated as long as the game continues;

**Punishment Phase**: If there was a deviation from $\sigma$, play $(D, D)$ and Player 1 uses an optimal exit strategy given that $(D, D)$ is repeated as long as the game continues.

LEMMA 8. *If $g - c < d - \ell$ and $c(g - c) \leqq (d - \ell)(c - v)$, then $\underline{\delta}(c, c) \leqq \underline{\delta}(\mathbf{u})$ for any $\mathbf{u} \in F$ such that $u_1 > v$ and $u_2 > 0$.*

Proof of Lemma 8: Fix any $u \in F$. Depending on how $u$ locates in $F$, the *necessary* action profiles to be played in $\sigma$ are different. From Figure 8 we can see that:

(i) To attain a payoff combination in Area (i) on average, a correlated action profile must include $(C, D)$ and $(D, D)$ and either $(C, C)$ or $(D, C)$.

(ii) To attain a payoff combination in Area (ii) on average, a correlated action profile must include $(C, D)$ and $(C, C)$ and either $(D, D)$ or $(D, C)$.

(iii) To attain a payoff combination in Area (iii) on average, a correlated action profile must include $(C, C)$ and $(D, C)$ and either $(C, D)$ or $(D, D)$.

(iv) To attain a payoff combination in Area (iv) on average, a correlated action profile must include $(D, D)$ and $(D, C)$ and either $(C, C)$ or $(C, D)$.

We thus derive sufficient conditions for $(C, C)$, $(D, C)$, and $(C, D)$ to be followed and then apply them for each Area to determine the minimum sufficient $\delta$.

Let $(u_1, u_2)$ be the one-shot average payoff of a correlated action profile on the play path. In each period, one of the pure action profiles in the support gets to be realized.

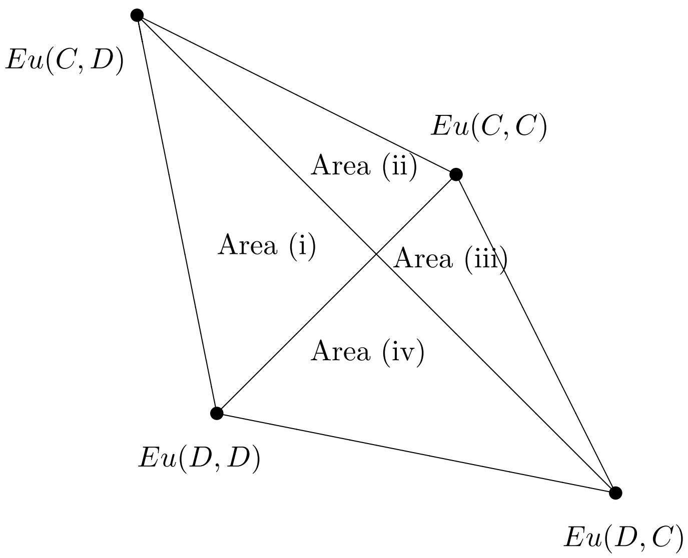



Figure 8: Areas in $F$

If $(C,C)$ is supposed to be played in this period, and if Player 1 deviates to $D$, her long-run payoff is $g+\delta v/(1-\delta)$ since they move to the punishment phase. If she follows $(C,C)$, the long-run payoff is $c+\delta u_1/(1-\delta)$, since the expected one-shot payoff from tomorrow on is $u_1$. Hence Player 1 does not deviate from $(C,C)$ if and only if

$$\begin{aligned} & c+\delta\frac{u_1}{1-\delta} \geqq g+\delta\frac{v}{1-\delta} \\ \iff \quad & \delta \geqq \frac{g-c}{(g-c)+(u_1-v)} =: \delta_1^{CC}(u_1). \end{aligned} \tag{23}$$

Similarly, Player 2 does not deviate from $(C,C)$ if and only if

$$\begin{aligned} & c+\delta\frac{u_2}{1-\delta} \geqq g+\delta\cdot 0 \\ \iff \quad & \delta \geqq \frac{g-c}{(g-c)+u_2} =: \delta_2^{CC}(u_2). \end{aligned} \tag{24}$$

When $(D,C)$ is supposed to be played, only Player 2 has an incentive to deviate. He does not deviate from $(D,C)$ if and only if

$$\begin{aligned} & \ell+\delta\frac{u_2}{1-\delta} \geqq d+\delta\cdot 0 \\ \iff \quad & \delta \geqq \frac{d-\ell}{(d-\ell)+u_2} =: \delta^{DC}(u_2). \end{aligned} \tag{25}$$

When $(C,D)$ is supposed to be played, only Player 1 has an incentive to deviate. She does not deviate from $(C,D)$ if and only if

$$\begin{aligned} & \ell + \delta\frac{u_1}{1-\delta} \geqq d + \delta\frac{v}{1-\delta} \\ \iff \quad & \delta \geqq \frac{d-\ell}{(d-\ell)+(u_1-v)} =: \delta^{CD}(u_1). \end{aligned} \tag{26}$$

In order to make these lower bounds less than 1, clearly we need $u_1 > v$ and $u_2 > 0$.

Note that in general, $f(x) = x/(A+x)$ is an increasing function of $x$ if and only if $A > 0$.[20] Therefore, if $g - c < d - \ell$, then $\delta_1^{CC}(u_1) < \delta^{CD}(u_1)$ and $\delta_2^{CC}(u_2) < \delta^{DC}(u_2)$ hold simultaneously. Note also that $v_2 < v_1$ implies that if $u_1 = u_2$, then $\delta_1^{CC}(u_1) > \delta_2^{CC}(u_2)$.

In order to make the players play a pure action profile $(C,C)$, we need $\delta \geqq \max\{\delta_1^{CC}(u_1), \delta_2^{CC}(u_2)\} =: \delta^{CC}(u)$. For players to play $(D,C)$ (resp. $(C,D)$), we only need $\delta \geqq \delta^{DC}(u_2)$ (resp. $\delta \geqq \delta^{CD}(u_1)$). Similarly, for correlated action profiles, we can classify the lower bound of $\delta$ as follows.

(i) In order to sustain $(u_1,u_2)$ in Area (i) with as small $\delta$ as possible, we must have at least $\delta \geqq \delta^{CD}(u_1)$ but also can use either $(C,C)$ or $(D,C)$ in the support of the correlated action profile. Hence the lowest $\delta$ is $\max\{\delta^{CD}(u_1), \min\{\delta^{CC}(u), \delta^{DC}(u_2)\}\}$. Under the assumption of $g - c < d - \ell$, this lower bound is equal to $\delta^{CD}(u_1)$.

Now, in order to lower $\delta^{CD}(u_1)$ as much as possible in this area, we must increase $u_1$ as large as possible, which hits the boundary with all other areas (see Figure 8). From Figure 8 it is easy to see that we cannot increase $u_1$ as much in Area (i) as in Area (ii), i.e.,

$$\min_{u_1 \in Area\ (i)} \delta^{CD}(u_1) > \min_{u_1 \in Area\ (ii)} \delta^{CD}(u_1).$$

(ii) In order to sustain $(u_1,u_2)$ in Area (ii), we need $\delta \geqq \max\{\delta^{CD}(u_1), \delta^{CC}(u)\}$, since for this we can ignore $(D,C)$ and use $(D,D)$ instead in the support, which does not require a high $\delta$. Under the assumption of $g - c < d - \ell$, this lower bound is $\delta^{CD}(u_1)$.

Again, in order to reduce $\delta^{CD}(u_1)$ as much as possible, we hit the boundary, which is $(C,C)$.

[20] By differentiation, $f'(x) = A/(A+x)^2$.

In summary so far, among the payoff combinations in the Area (i) and (ii), $(C, C)$ is the easiest to sustain. This is because in these areas, only Player 1's deviation must be prevented and $(C, C)$ gives the highest on-path average payoff for Player 1 in these areas.

By contrast, in Areas (iii) and (iv), we need to prevent Player 2's deviation so that the sufficient $\delta$s are as follows.

(iii) For $(u_1, u_2)$ in Area (iii), we need $\delta \geqq \max\{\delta^{DC}(u_2), \delta^{CC}(u)\} = \delta^{DC}(u_2)$.

(iv) For $(u_1, u_2)$ in Area (iv), we need $\delta \geqq \max\{\delta^{DC}(u_2), \min\{\delta^{CC}(u), \delta^{CD}(u_1)\}\} = \delta^{DC}(u_2)$.

To reduce $\delta^{DC}(u_2)$ as much as possible, we should increase $u_2$. Hence the minimum $\delta^{DC}(u_2)$ is attained in Area (iii) where $u_2 = c$. Recall that $\delta_1^{CC}(c) > \delta_2^{CC}(c)$ since $v_2 < v_1$. Hence $(C, C)$ is the easiest to sustain in Area (iii) and (iv) if $\delta^{DC}(c) \geqq \delta_1^{CC}(c)$. This is equivalent to

$$\frac{d-\ell}{(d-\ell)+c} \geqq \frac{g-c}{(g-c)+(c-v)} \iff (d-\ell)(c-v) \geqq c(g-c).$$

Therefore, we have that if $g - c < d - \ell$ and $(d - \ell)(c - v) \geqq c(g - c)$, then $\delta^D(c, c) \leqq \delta^D(u)$ for any $u \in F$ such that $u_1 > v$ and $u_2 > 0$. □

## APPENDIX B: Proofs

**Proof of Proposition 2**: **Case 1:** Assume that $c \geqq v + \epsilon$. Recall that

$$\frac{c}{1-\delta} \geqq g + \delta\frac{v}{1-\delta} \iff \delta \geqq \delta^D(v) \tag{2}$$

$$g + \delta W \geqq g + \delta\frac{v}{1-\delta} \iff \delta \geqq \delta^P(v, \epsilon) \tag{6}$$

We also show that the on-path value function $c/(1 - \delta)$ exceeds the deviation value $g + \delta W$ for any $\delta$ above some critical $\delta$. By computation,

$$\begin{aligned}
& \frac{c}{1-\delta} \geqq g + \delta W, \\
& \frac{c}{1-\delta} \geqq g + \delta\frac{\frac{v+\epsilon}{1-\delta} + d}{2-\delta}, \\
\iff \quad & (2-\delta)c \geqq (1-\delta)(2-\delta)g + \delta(v+\epsilon) + \delta(1-\delta)d, \\
\iff \quad & h(\delta) := -\delta^2(g-d) + \delta\{3g - (v+\epsilon) - c - d\} - 2(g-c) \geqq 0.
\end{aligned}$$

Notice that $h$ is quadratic in $\delta$, $h(0) = -2(g-c) < 0$ and $h(1) = c - (v+\epsilon) \geqq 0$. Therefore there exists $\delta^{cW}(v,\epsilon) \in (0,1]$ such that for any $\delta \geqq \delta^{cW}(v,\epsilon)$, $h(\delta) \geqq 0$ holds. Thus,

$$\frac{c}{1-\delta} \geqq g + \delta W \iff \delta \geqq \delta^{cW}(v,\epsilon). \tag{27}$$

Note also that $\delta^D(v) \leqq \delta^P(v,\epsilon)$ if and only if $\epsilon \leqq (v-d)(c-v)/(g-v)$. Now we divide the analysis into two cases.

Case 1-a: $0 < \epsilon \leqq (v-d)(c-v)/(g-v)$, i.e., $\delta^D(v) \leqq \delta^P(v,\epsilon)$.

In this case, the on-path value function $\frac{c}{1-\delta}$ intersects with $g + \delta\frac{v}{1-\delta}$ at $\delta^D(v)$ and at that point $\frac{v}{1-\delta} > W$. Hence (27) implies that $c/(1-\delta)$ exceeds $g + \delta W$ for any $\delta \geqq \delta^D(v)$. Therefore $\frac{c}{1-\delta} \geqq \max\{g+\delta W, g+\delta\frac{v}{1-\delta}\}$ if and only if $\delta \geqq \delta^D(v)$. See Figure 2(a).

Player 2's deviation value changes depending on whether Player 1 exits immediately or not after seeing a deviation. If Player 1 exits immediately, i.e., if $\max\{W, \frac{v}{1-\delta}\} = \frac{v}{1-\delta}$, Player 2's deviation value is $g + \delta \cdot 0$. In this case $\delta \geqq \delta^D(v)$ implies that $c/(1-\delta) > g$ so that Player 2's deviation is prevented.

If Player 1 waits for the good option in the punishment phase, i.e., if $\max\{W, \frac{v}{1-\delta}\} = W$, then Player 2's deviation value is increased to

$$g + \frac{\delta}{2}d + (\frac{\delta}{2})^2 d + \cdots = g + \frac{\delta d}{2-\delta}.$$

In this case Player 2 does not deviate in the cooperation phase if and only if

$$\begin{aligned} & \frac{c}{1-\delta} \geqq g + \frac{\delta d}{2-\delta} \\ \iff \quad & h'(\delta) := -\delta^2(g-c) + \delta(3g-c-d) - 2(g-c) \geqq 0. \end{aligned} \tag{28}$$

This $h'$ has the property that once it exceeds 0 at some $\delta$, $h'(\delta) \geqq 0$ for all larger $\delta$. Plug in $\delta^D(v)$ and we get

$$h'(\delta^D(v)) = \frac{(g-c)}{(g-v)^2}\{(c-v)(v-d) + v(g-v)\} > 0.$$

Therefore for any $\delta \geqq \delta^D(v)$, (28) is satisfied. Note that this argument for Player 2 does not rely on the assumption that $\delta^D(v) \leqq \delta^P(v,\epsilon)$.

In sum, when $\delta^D(v) \leqq \delta^P(v,\epsilon)$, the eternal cooperation is sustained if and only if $\delta \geqq \delta^D(v)$, that is $\delta^S(v,\epsilon) = \delta^D(v)$.

Case 1-b: $(v-d)(c-v)/(g-v) < \epsilon$, i.e., $\delta^P(v,\epsilon) < \delta^D(v)$.

In this case, when the on-path value $c/(1-\delta)$ intersects with $g + \delta\frac{v}{1-\delta}$ (at $\delta^D(v)$), the optimal one-shot deviation value is in fact $g+\delta W$. Thus the on-path value function intersects with the optimal one-shot deviation value $\max\{g+\delta W, g+\delta\frac{v}{1-\delta}\}$ when the latter is $g+\delta W$ (see Figure 2(b)), at $\delta^{cW}(v,\epsilon)$. At this point $v/(1-\delta) < W$. Therefore from (2) and (27),

$$\frac{c}{1-\delta} \geqq \max\{g+\delta W, g+\delta\frac{v}{1-\delta}\} \iff \delta \geqq \delta^{cW}(v,\epsilon).$$

We show that $\delta^{cW}(v,\epsilon) > \delta^D(v)$. As we have seen,

$$h(\delta^D(v)) = \frac{(g-c)}{(g-v)^2}\{(c-v)(v-d) - \epsilon(g-v)\}.$$

Therefore for $(v-d)(c-v)/(g-v) < \epsilon$, $h(\delta^D(v)) < 0$. This means that $\delta^{cW}(v,\epsilon)$, above which $h(\delta) \geqq 0$, must be strictly greater than $\delta^D(v)$.

Since Player 2 does not deviate in the cooperation phase if $\delta \geqq \delta^D(v)$, we conclude that when $\delta^P(v,\epsilon) < \delta^D(v)$, the eternal cooperation is sustained if and only if $\delta \geqq \delta^{cW}(v,\epsilon)$.

**Case 2:** Suppose that $v+\epsilon > c$.

First, we show that there exists a unique $\delta^V(v,\epsilon) \in (0,\delta^D(v))$ such that for any $\delta \geqq \delta^V(v,\epsilon)$ (see Figure 3),

$$c+\delta V \geqq g+\delta\frac{v}{1-\delta}.$$

Let

$$\begin{aligned} h(\delta,v,\epsilon) &:= (1-\delta)(2-\delta)\{c+\delta V - g - \delta\frac{v}{1-\delta}\} \\ &= -(g-v)\delta^2 + \{3g-2c-(v-\epsilon)\}\delta - 2(g-c). \end{aligned}$$

Then

$$c+\delta V \geqq g+\delta\frac{v}{1-\delta} \iff h(\delta,v,\epsilon) \geqq 0.$$

Since $h(\delta,v,\epsilon)$ is a concave, quadratic function of $\delta$, $h(0,v,\epsilon) = -2(g-c) < 0$, and $h(1,v,\epsilon) = \epsilon > 0$, there exists a unique $\delta^V(v,\epsilon) \in (0,1)$ such that for any $\delta \geqq \delta^V(v,\epsilon)$, $h(\delta,v,\epsilon) \geqq 0$ holds. To show that $\delta^V(v,\epsilon) < \delta^D(v)$, plug in $\delta = \delta^D(v)$ into $h$ and we get

$$h(\delta^D(v),v,\epsilon) = \frac{(g-c)(v+\epsilon-c)}{g-v} > 0.$$

Thus $\delta^D(v) > \delta^V(v, \epsilon)$.

Second, we show that Player 1 does not deviate for any $\delta \geqq \max\{\delta^V(v, \epsilon), \frac{2(g-c)}{g-d}\}$. Recall that from (8), we have that

$$c + \delta V \geqq g + \delta W \iff \delta \geqq \frac{2(g-c)}{g-d}.$$

Note that

$$\frac{2(g-c)}{g-d} < \delta^D(v) \iff g + d < 2v. \tag{29}$$

Therefore

$$c + \delta V \geqq \max\{g + \delta W, g + \delta \frac{v}{1-\delta}\}$$

for any $\delta \geqq \max\{\delta^V, \frac{2(g-c)}{g-d}\}$.

Next, consider Player 2. Let $V_2$ be the continuation payoff during the cooperation phase for Player 2. Since Player 1 exits with probability 1/2, it satisfies

$$V_2 = \frac{1}{2}\{c + \delta V_2\} + \frac{1}{2} \cdot 0.$$

Thus $V_2 = c/(2-\delta)$ and the on-path value for Player 2 is $c + \delta V_2 = \frac{c}{1-\delta/2}$.

If he deviates, Player 1 exits immediately if $v/(1-\delta) \geqq W$ or equivalently $\delta \leqq \delta^P(v, \epsilon)$, and Player 1 waits for the good option otherwise. Let $W_2$ be the continuation payoff during the punishment phase for Player 2, when Player 1 waits for the good option. It satisfies

$$W_2 = \frac{1}{2}\{d + \delta W_2\} + \frac{1}{2} \cdot 0,$$

so that $W_2 = d/(2-\delta)$. Hence the one-shot deviation value for Player 2 is

$$\begin{cases} g + \delta \cdot 0 & \text{if } \delta \leqq \delta^P(v, \epsilon) \\ g + \delta W_2 & \text{if } \delta^P(v, \epsilon) \leqq \delta. \end{cases}$$

Since $d \geqq 0$, it suffices to show that the lower bound of $\delta$ that satisfies

$$c + \delta V_2 \geqq g + \delta W_2$$

is less than $\delta^D(v)$. Note that the payoff structure is similar for Player 2 and Player 1;

$$V_2 - W_2 = \frac{1}{2}\{c + \delta V_2\} + \frac{1}{2} \cdot 0 - \frac{1}{2}\{d + \delta W_2\} - \frac{1}{2} \cdot 0,$$

and

$$V - W = \frac{1}{2}\{c + \delta V\} + \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta} - \frac{1}{2}\{d + \delta W\} - \frac{1}{2} \cdot \frac{v+\epsilon}{1-\delta}.$$

Hence $V_2 - W_2 = V - W$ and since

$$c + \delta V \geqq g + \delta W \iff \delta \geqq \frac{2(g-c)}{g-d},$$

Player 2 does not deviate if and only if $\delta \geqq \frac{2(g-c)}{g-d}$.

Therefore $\delta^S(v, \epsilon) = \max\{\delta^V(v, \epsilon), \frac{2(g-c)}{g-d}\}$ is the lower bound of the discount factor that sustains the stochastic cooperation. Finally, note that $\delta^S(v, \epsilon) < \delta^D(v)$ if and only if $v > (g + d)/2$. □

**Proof of Proposition 3**: The proof is essentially analogous to that of Proposition 2. Case 1: First we show that there exists $\delta^{cW}(p) \in (0, 1]$ such that

$$\frac{c}{1-\delta} \geqq g + \delta W(p) \iff \delta \geqq \delta^{cW}(p). \tag{30}$$

By computation,

$$\begin{aligned}
& \frac{c}{1-\delta} \geqq g + \delta W(p) \\
\iff & \{c - g(1-\delta)\}\{1 - (1-p)\delta\} \geqq \delta p v^+ + \delta(1-p)(1-\delta)d \\
\iff & h_p(\delta) := -(1-p)(g-d)\delta^2 + \delta\{(1-p)(g-c) + g - (1-p)d - pv^+\} - (g-c) \geqq 0.
\end{aligned}$$

Again $h_p(\cdot)$ is a concave function of $\delta$, $h_p(0) = -(g-c) < 0$, and $h_p(1) = p(c - v^+) \geqq 0$ for any $p \in (0, 1)$. Therefore there exists a unique $\delta^{cW}(p) \in (0, 1]$ such that (30) holds.

From (12), we also have

$$W(p) \geqq \frac{v}{1-\delta} \iff \delta \geqq \frac{v^- - d}{v - d}.$$

Thus, depending on whether $\delta^D(v) \leqq \frac{v^- - d}{v-d}$ or $\frac{v^- - d}{v-d} < \delta^D(v)$, we have slightly different arguments. When $\delta^D(v) \leqq \frac{v^- - d}{v-d}$, the on-path value function $c/(1-\delta)$ intersects with the optimal one-shot deviation value at $\delta^D(v)$, since $v/(1-\delta) > W(p)$ at $\delta^D(v)$. Hence in this case Player 1 does not deviate if and only if $\delta \geqq \delta^D(v)$. When $\frac{v^- - d}{v-d} < \delta^D(v)$, the on-path value function intersects with the optimal one-shot deviation value at $\delta^{cW}(p)$. Let us show that $\delta^{cW}(p) > \delta^D(v)$. Using $v = pv^+ + (1-p)v^-$, we have

$$h_p(\delta^D(v)) = \frac{(1-p)(g-c)}{(g-v)^2}[(g-v)(v^- - d) - (g-c)(v-d)] < 0$$

since $\frac{v^- - d}{v-d} < \frac{g-c}{g-v} = \delta^D(v)$. Thus $\delta^D(v) < \delta^{cW}(p)$.

For Player 2, his deviation value is either $g + \delta \cdot 0$ when $\delta < \frac{v^- - d}{v - d}$ so that Player 1 exits immediately after a deviation or $g + \delta W_2(p)$ where $W_2(p)$ satisfies

$$W_2(p) = p \cdot 0 + (1 - p)\{d + \delta W_2(p)\},$$

when $\delta \geqq \frac{v^- - d}{v - d}$ so that Player 1 waits for $v^+$ in the punishment phase. In the former case, $\delta \geqq \delta^D(v)$ implies that $c/(1 - \delta) > g$, hence Player 2 does not deviate. In the latter case, notice that $W(p) > W_2(p)$ since $v^+ > d \geqq 0$. Therefore Player 1's condition $c/(1 - \delta) \geqq g + \delta W(p)$ implies that Player 2 does not deviate either.

In summary, the eternal cooperation is sustained if and only if $\delta \geqq \delta^E(p) := \max\{\delta^D(v), \delta^{cW}(p)\}$ and $\delta^E(p) \geqq \delta^D(v)$.

Case 2: Let

$$\begin{aligned} h'_p(\delta) &:= (1 - \delta)\{1 - (1 - p)\delta\}\{c + \delta V(p) - g - \delta \frac{v}{1 - \delta}\} \\ &= -(1 - p)(g - v)\delta^2 + \{-p(g - v^+) + 2g - c - v\}\delta - (g - c). \end{aligned}$$

Then

$$c + \delta V(p) \geqq g + \delta \frac{v}{1 - \delta} \iff h'_p(\delta) \geqq 0.$$

Since $h'_p(\delta)$ is a concave, quadratic function of $\delta$, $h'_p(0) = -(g - c) < 0$, and $h'_p(1) = p(v^+ - v) > 0$, there exists a unique $\delta^V(p) \in (0, 1)$ such that for any $\delta \geqq \delta^V(p)$, $h'_p(\delta) \geqq 0$ holds. To show that $\delta^V(p) < \delta^D(v)$, plug in $\delta^D(v)$ into $h'_p$ and we obtain

$$h'_p(\delta^D(v)) = \frac{(g - c)(v^+ - c)}{g - v} > 0.$$

Hence $\delta^V(p) < \delta^D(v)$.

Recall that $c + \delta V(p) \geqq g + \delta W(p)$ if and only if $\delta \geqq \frac{g - c}{(1 - p)(g - d)}$ and this is less than $\delta^D(v)$ under the assumption of (16). Therefore Player 1 does not deviate if and only if $\delta \geqq \max\{\delta^V(p), \frac{g - c}{(1 - p)(g - d)}\} =: \delta^S(p)$ and this bound is less than $\delta^D(v)$ if and only if $p < (v - d)/(g - d)$.

Next consider Player 2. Let $V_2(p)$ be the continuation value during the cooperation phase for Player 2. Since Player 1 exits with probability $p$, it satisfies

$$V_2(p) = (1 - p)\{c + \delta V_2(p)\} + p \cdot 0.$$

The on-path value for Player 2 is $c + \delta V_2(p)$. Similarly, let $W_2(p)$ be the continuation payoff during the punishment phase for Player 2, when Player 1 waits for the good option. It satisfies

$$W_2(p) = (1 - p)\{d + \delta W_2(p)\} + p \cdot 0.$$

Thus the one-shot deviation value for Player 2 is

$$\begin{cases} g+\delta\cdot 0 & \text{if } \frac{v}{1-\delta} \geqq V(p) \\ g+\delta W_2(p) & \text{if } V(p) \geqq \frac{v}{1-\delta}. \end{cases}$$

Since $d \geqq 0$, it suffices to show that the lower bound of $\delta$ that satisfies

$$c + \delta V_2(p) \geqq g + \delta W_2(p)$$

is less than $\delta^D(v)$. As in the case of the 1/2-binary distribution, the payoff structure is similar for Player 2 and Player 1;

$$V(p) - W(p) = (1-p)\{c+\delta V(p)\} + p\frac{v^+}{1-\delta} - (1-p)\{d+\delta W(p)\} - p\frac{v^+}{1-\delta}$$

while

$$V_2(p) - W_2(p) = (1-p)\{c+\delta V_2(p)\} - (1-p)\{d+\delta W_2(p)\}.$$

Hence

$$V(p) - W(p) = V_2(p) - W_2(p)$$

and thus

$$c + V_2(p) \geqq g + \delta W_2(p) \iff \delta \geqq \frac{g-c}{(1-p)(g-d)}.$$

In summary, both players do not deviate if and only if $\delta \geqq \delta^S(p) = \max\{\delta^V(p), \frac{g-c}{(1-p)(g-d)}\}$. Finally, note that $p < (v-d)/(g-d)$ if and only if $\delta^S(p) < \delta^D(v)$. □

**Proof of Lemma 6**: From (17), we have

$$W' = \frac{2(v+\epsilon) + (v-\epsilon) + (1-\delta)d}{(1-\delta)(4-\delta)}.$$

By computation

$$\begin{aligned} (\frac{v}{1-\delta} - W')(1-\delta)(4-\delta) &= (4-\delta)v - 2(v+\epsilon) - (v-\epsilon) - (1-\delta)d \\ &= -\delta(v-d) + v - d - \epsilon, \end{aligned}$$

so that

$$\frac{v}{1-\delta} \geqq W' \iff \delta \leqq \delta^P(v,\epsilon). \tag{31}$$

Moreover, by comparing (5) and (17);

$$\begin{aligned} W &= \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta}+\frac{1}{4}(d+\delta W)+\frac{1}{4}(d+\delta W) \\ W' &= \frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta}+\frac{1}{4}\cdot\frac{v-\epsilon}{1-\delta}+\frac{1}{4}(d+\delta W') \\ \Rightarrow W-W' &= \frac{d+\delta W-\frac{v-\epsilon}{1-\delta}}{4-\delta}. \end{aligned}$$

Therefore $W \geqq W'$ if and only if

$$\begin{aligned} & d+W \geqq \frac{v-\epsilon}{1-\delta} \\ \iff\quad & W=\frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta}+\frac{1}{2}(d+\delta W)\geqq\frac{1}{2}\cdot\frac{v+\epsilon}{1-\delta}+\frac{1}{2}\frac{v-\epsilon}{1-\delta}=\frac{v}{1-\delta} \\ \iff\quad & \delta \geqq \delta^P(v,\epsilon). \end{aligned}$$

Combined with (31), we have that

$$\begin{aligned} \delta \leqq \delta^P(v,\epsilon) \quad &\Rightarrow\quad g+\delta\frac{v}{1-\delta}\geqq g+\delta W'\geqq g+\delta W; \\ \delta^P(v,\epsilon)\leqq\delta \quad &\Rightarrow\quad g+\delta W\geqq g+\delta W'\geqq g+\delta\frac{v}{1-\delta}. \end{aligned}$$

□

**Proof of Proposition 6**: It suffices to prove that Player 2 does not deviate under $\delta \geqq \delta^F$. Recall that Player 1 exits with probability $1-F(r^*(d,\delta))$ if the optimal reservation level is $r^*(d,\delta)$. Hence Player 2's deviation value is

$$\begin{cases} g+\delta\cdot 0 & \text{if } \delta\leqq\frac{v-d}{v-d} \\ g+\delta\frac{d}{1-\delta F(r^*(d,\delta))} & \text{if } \frac{v-d}{v-d}\leqq\delta. \end{cases}$$

Player 2's total expected payoff in the cooperation phase is

$$\frac{c}{1-\delta F(r^*(c,\delta))}.$$

Since we have assumed that $\delta^D(v)\leqq\frac{v-d}{v-d}$, it suffices to show that the smallest $\delta$ that satisfies

$$\frac{c}{1-\delta F(r^*(c,\delta))}\geqq g \tag{32}$$

is not more than $\delta^D(v)$. By rearrangement, (32) is equivalent to

$$\delta F(r^*(c,\delta))g \geqq g-c.$$

We first prove that $c < r^*(c,\delta)$. Notice that $\overline{v} > c$ is equivalent to

$$\begin{aligned} & \int_c^{\overline{v}} (x-c)f(x)dx > 0 \\ \Longleftrightarrow \quad & \int_c^{\overline{v}} xf(x)dx + F(c)c > c \\ \Longleftrightarrow \quad & (1-\delta)c + \delta \int_c^{\overline{v}} xf(x)dx + \delta F(c)c > c. \end{aligned}$$

This implies that at $r = c$, the RHS of (21) is above the 45 degree line. Hence the intersection with the 45 degree line (which is $r^*(c,\delta)$) is greater than $c$ for any $\delta$. (See Figure 7.) Therefore we also have that $F(c) < F(r^*(c,\delta))$ for any $\delta$, and thus $v > \{1 - F(c)\}g$ implies that

$$\delta F(r^*(c,\delta))g > \delta F(c)g > \delta(g-v).$$

Second, note that when $\delta = \delta^D(v)$, $\delta(g-v) = g-c$. Therefore at $\delta = \delta^D(v)$,

$$\delta F(r^*(c,\delta))g > g-c,$$

and $\delta F(r^*(c,\delta))g$ is uniformly greater than $\delta F(c)g$ for any $\delta \in (0,1)$. Thus there exists $\delta^{F2} < \delta^D(v)$ such that for any $\delta \geqq \delta^{F2}$, Player 2 does not deviate. Let $\delta^{F1}$ be the bound for Player 1, then as shown in the text $\delta^{F1} < \delta^D(v)$ as well. Let $\delta^F = \max\{\delta^{F1}, \delta^{F2}\}$ then this is the lower bound that sustains the stochastic cooperation and is strictly smaller than $\delta^D(v)$. □

## REFERENCES

ABREU, D. (1988): "On the Theory of Infinitely Repeated Games with Discounting," *Econometrica*, 56, 383-396.

BAYE, M., and D. JANSEN (1996): "Repeated Games with Stochastic Discounting," *Economica*, 63, 531-541.

DAL BÓ, P. (2007): "Tacit Collusion under Interest Rate Fluctuations," *RAND Journal of Economics*, 38, 1-8.

DATTA, S. (1996): "Building Trust," Working Paper, London School of Economics.

DUTTA, P. (1995): "A Folk Theorem for Stochastic Games," *Journal of Economic Theory*, 66, 1-32.

ELLISON, G. (1994): "Cooperation in the Prisoner's Dilemma with Anonymous Random Matching," *Review of Economic Studies*, 61, 567-588.

FUJIWARA-GREVE, T. (2002): "On Voluntary and Repeatable Partnerships under No Information Flow," *Proceedings of the 2002 North American Summer Meetings of the Econometric Society* (http://www.dklevine.com/proceedings/game-theory.htm).

FUJIWARA-GREVE, T. and M. OKUNO-FUJIWARA (2009): "Voluntary Separable Repeated Prisoner's Dilemma," *Review of Economic Studies*, 76, 993-1021.

FURUSAWA, T., and T. KAWAKAMI (2008): "Gradual Cooperation in the Existence of Outside Options," *Journal of Economic Behavior and Organization*, 68, 378-389.

GHOSH, P., and D. RAY (1996): "Cooperation in Community Interaction without Information Flows," *Review of Economic Studies*, 63, 491-519.

KANDORI, M. (1992): "Social Norms and Community Enforcement," *Review of Economic Studies*, 59, 63-80.

KRANTON, R. (1996a): "The Formation of Cooperative Relationships," *Journal of Law, Economics, & Organization* 12, 214-233.

———- (1996b): "Reciprocal Exchange: A Self-Sustaining System," *American Economic Review*, 86, 830-851.

MAILATH, G., and L. SAMUELSON. (2006) *Repeated Games and Reputations.* Oxford University Press.

OKUNO-FUJIWARA, M., and A. POSTLEWAITE (1995): "Social Norms and Random Matching Games," *Games and Economic Behavior*, 9, 79-109.

ROTEMBERG, J., and G. SALONER (1986): "A Supergame-Theoretic Model of Price Wars during Booms," *American Economic Review*, 76, 390-407.

WATSON, J. (2002): "Starting Small and Commitment," *Games and Economic Behavior*, 38, 176-199.

YASUDA, Y. (2007): "The Theory of Collusion under Financial Constraints," manuscript, GRIPS.

YASUDA, Y., and T. FUJIWARA-GREVE (2009): "Cooperation in Repeated Prisoner's Dilemma with Perturbed Payoffs," manuscript, GRIPS and Keio University. (Available at http://ssrn.com/abstract=1420822.)