Inequality and the Social Stability of Economies with

Collective Property Rights

Gerhard Glomm* and Roger D. Lagunoff†

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Abstract

In this paper, a model of an economy with collective property rights defined over some commodities is considered. We investigate what role, if any, income inequality plays in providing social stability in the sense that the economy exhibits a nonempty core. It is shown through a series of examples that reductions of income inequality can play a crucial role in providing social stability to the extent that these reductions increase "specialization" in the economy. We give a general result to this effect. It is also shown that our notion of specialization is not limited to the privatized sector of the economy. Even in economies in which there is no privately held property, sufficient specialization guarantees that the economy is socially stable.

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1   Introduction

Much of formal economic theorizing assumes that resources are owned privately, that is by individuals. In Debreu (1959), for example, resource endowments as well as claims to profit shares are attributed to individual consumers. Instances in which legal rights to resources are held by coalitions of agents rather than by individuals are, however, numerous. In places as diverse as England, Japan, and Russia, collective property rights over commodities such as grazing and agricultural land, workshops, mills, water, and fishing rights were considered the norm historically.[3] Swiss Alp territory is still owned by villages which collectively make decisions about grazing, fertilization, maintenance etc. In the U.S. legal partnerships are owned by coalitions, the partners, and not by individuals. Specific rules are used to make joint allocation decisions concerning the jointly held assets. Some instances of coalitional property rights have lasted for centuries, while in other instances such coalitional rights were short lived.

It is often claimed that collective property rights regimes are less stable than private property regimes. The argument runs along the lines of showing that without private owner-ship that there is a potential Prisoners' Dilemma type of problem in which each individual,

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[3]See Dahlman (1980) and McFarlane (1978) on the English open field system, Troost (1990) and Brown (1990) on the Japanese village communities, and Shanin (1972) on the Russian mir (peasant communes).

in anticipating the resource usage of others, tends to "overuse" the resource over which ownership rights are jointly held. The resource is therefore depleted more quickly than under private ownership.[4] It is also sometimes claimed that collective property regimes are more egalitarian than private property regimes This is sometimes attributed to the more equal access under democratic "political" allocation mechanisms than "economic" ones.[5] In this paper we investigate what role, if any, income inequality plays in providing social stability of collective property rights regimes.

Specifically, we consider a model of an exchange economy which in many ways is similar to the one set forth by Debreu (1959) but in which the legal framework assigns endowments of some commodities to coalitions and not only to individual consumers.

We attempt to capture the notion of social stability by considering the *core* of this economy that exhibits collectively held property. The use of the core as a stability concept allows us to investigate the conditions under which collective property rights regimes are stable in the sense that there are outcomes against which no coalition in the economy has incentives to "recontract out".

The model is described formally in section 2. The model considered here is a special case of the general model of *coalitional property rights (CPR) regimes* of Glomm and Lagunoff (1992). In that paper ownership rights to various resources are held by groups of various sizes rather than by single individuals or all of society. Certain conditions on both the primitives and the legal structure were shown to give rise to outcomes against which no

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[4]See Ostrom (1989) for a summary of the enormous "tragedy of the commons" literature.

[5]This is the substance of J. K. Galbraith's argument (1952).

group in society would attempt to "recontract out", i.e., give rise to outcomes that are in the core of the correponding CPR regime.

By contrast, in this paper we consider only the two polar cases of individual and pure collective ownership. Here, as before, we hope to shed some light on why some property rights regimes last while others do not. In this paper, however, we concentrate on the role of income inequality and its connection to stability.

Section 3 demontrates the basic stability problem with collective ownership. When allocation decision procedures for collectively held goods deny individuals the power of veto, the core is often empty. This is best demonstrated with majority rule procedures, though it is not limited to majority rule. Majority rule creates a patterns similar to Condorcet voting cycles that destabilize majority voting procedures.

In section 4 we consider a set of cases to demonstrate the connection between inequality of incomes and non-emptiness of the core. We present some simple examples to highlight our results.

When all goods are communally owned and when allocation decisions within coalitions are made through majority rule we find that the core is empty. This is also the case when exactly one good is privately owned, all others are held by coalitions, and coalitional decisions are made through majority rule. Here the distribution of the private endowment and hence the inequality of income is irrelevant for the emptiness of the core. These economies are socially unstable.

When there are many privately held goods we find that specialization of endowments as well as inequality matters. In fact, non-emptiness of the core requires sufficient *specialization*

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in endowments holdings, a factor that also reduces income inequality.

Section 5 extends the analysis to "pure" collective property rights economies. We show that the notions of specialization and inequality is not limited to private property. Even economies without private property may be socially stable if there is a sufficient amount of "political" specialization. What ultimately matters for stability is an individual's "veto power" which is determined by his marginal contribution to the economy. This contribution is not necessarily determined purely by one's private wealth.

Section 6 contains concluding remarks, and Section 7 is an Appendix that contains proofs of all the results.

As a cautionary note, we do not consider either production or public goods in this paper. The reason is that we wish to focus on the consequences of pure exchange for stability, and on the legal structure rather than on technological aspects exclusion and rivalry of certain goods.

2 Framework

2.1 The Primitives

In this section we describe a model of an economy in which there is collective property rights in some goods. We also introduce as our concept of social stability, the core. In our economy there is a finite set of individuals $I = \{1, \ldots, n\}$ with a single indvidual denoted by $i \in I$. There is a finite number of goods ($k = 1, \ldots, \ell$). For each good there is an aggregate endowment; we let $w = (w_k) \in Rplus^{\ell}$ denote the vector of aggregate endowments. We will call an allocation any vector $x = (x_k^i)_{i=1,\ldots,n,\ k=1,\ldots,\ell}$ which satisfies $\sum_{i \in I} x^i \leq w_k$ for all k.

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Each individual i has a utility function, $u^i : \Re_+^\ell \to \Re$. Let $\bar{u} = (u^i)_{i \in I}$. The following assumptions on indiviuals' utilities will be used at various points in the analysis.

(A1) All utility functions are assumed strictly concave and strictly increasing in each argument.

(A2) All agents have identical preferences, i.e., $u^i = u^j$ for all i, j.

(A3) Utility function u is symmetric across goods, i.e., for any permutation mapping $\phi : \{1, \ldots \ell\} \to \{1, \ldots \ell\}$, u satisfies $u(x_1, \ldots, x_\ell) = u(x_{\phi(1)}, \ldots, x_{\phi(\ell)})$.

(A4) For any $m < n < \ell$, $u : \Re_+^\ell \to \Re$ satisfies:

$$\lim_{x \to \infty} \left[u(\frac{x}{n}, \ldots, \frac{x}{n}) - u(\frac{x}{n-m}, \ldots, \frac{x}{n-m}, \overbrace{0, \ldots, 0}^{\ell-m}) \right] > 0. \qquad (1)$$

A few comments are in order concerning these assumptions. Assumption (A1) is standard and will be assumed throughout the the analysis. In this paper we wish to focus on income inequality and not preference heterogeneity. To facilitate cross-income comparisons, we will use the assumption of homogeneous preferences, (A2), thoughout much of the analysis. Assumption (A3) also facilitates cross-income comparisons when agents have heterogeneous endowment holdings.[6]

Assumption (A4) is a curvature condition. It states that any sufficiently large allocation

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[6]Many commonly used parameterizations satisfy (A3) such as CES utility of the form,

$$u = \left(\sum_{k=1}^{\ell} (x_k)^\varrho \right)^{\frac{1}{\varrho}}$$

where $-\infty < \varrho < 1$.

shared equally among all n individuals dominates the allocation resticted to a subset of $\ell - m$ goods shared equally by $n - m$ individuals.

2.2 The Property Rights Structure

We consider here a very simple extension of the typical Arrow-Debreu private property structure. We will assume that the first ℓ_1 goods are privately held by individuals, while the last $\ell_2 \equiv \ell - \ell_1$ goods are collectively held by the "grand coalition" I. In this paper we will typically let $\ell_1 = n$ (n, recall, is the total number of individuals). For each $k = 1, \ldots \ell_1$, let α_{ik} denote the endowment of good k privately held by the ith individual. Obviously we must have

$$\sum_{i \in I} \alpha_{ik} = w_k \ \forall k = 1, \ldots \ell_1.$$

For the final $\ell_2 = \ell - \ell_1$ goods, property rights are assigned to the grand coalition $I = \{1, \ldots, n\}$. We denote the endowment of each such good k by γ_k. We then have $\gamma_k = w_k$ for all $k = \ell_1 + 1, \ldots \ell$).

Since some goods are collectively owned in this model it is necessary to specify how collective resource allocation decisions are made. One reduced form method of doing this is to specify the subcoalitions in the economy that are *winning* with respect to the resource in question. A winning coalition is one that exercises complete control over the allocation of the resource. For example under *majority rule*, any subcoalition with at least 50% of the members of I can choose to allocate the candidate resource among themselves. In this case individuals in the minority can be completely excluded from consuming any of the

collectively held good.[7]

For each good k, $(k = \ell_1 + 1, \ldots \ell)$, we denote the set of *winning coalitions* by $\mathcal{D}_k$. A winning coalition in $\mathcal{D}_k$ will be denoted by d_k. If, for example, there are three agents, $I = \{1, 2, 3\}$, then given some good k under majority rule the winning coalitions are given by the set $\mathcal{D}_k = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Let $\mathcal{D} = (\mathcal{D}_k)$. The tuple $\mathcal{D}$ is sometimes referred to as the collection of "exclusion rules" (see Glomm and Lagunoff (1992)).

We define an economy with collective property rights and exclusion rules to be the tuple,

$$\mathcal{E} = (\bar{u}, w, \mathcal{D}) \tag{2}$$

where $\bar{u}$ is the list of individuals' utility functions, $w = (w_1, \ldots, w_{\ell_1}, w_{\ell_1+1}, \ldots, w_\ell)$ is the vector of endowments in which each of the first ℓ_1 entries is divided up into individually held pieces, and the last ℓ_2 entries are held collectively and allocated vis-a-vis exclusion rules $\mathcal{D}$.

2.3 The Core

Given any allocation for the economy $\mathcal{E}$, we ask if there exists a coalition of individuals that can, given the legal structure described by $\mathcal{E}$, improve upon the candidate allocation. Formally, we define an allocation x to be in the *core of $\mathcal{E}$* if there is no coalition $C \subseteq I$ and no coalitional allocation $(y^i)_{i \in C}$ that satisfies

$$
\begin{array}{l}
\text{if } k \leq \ell_1, \text{ then } \sum_{i \in C} y_k^i \leq \sum_{i \in C} \alpha_{ik} \\
\text{if } k > \ell_1, \text{ then } \sum_{i \in C} y_k^i \leq \begin{cases} \gamma_k & \text{if there exists some } d_k \in \mathcal{D}_k \text{ with } d_k \subseteq C \\ 0 & \text{otherwise} \end{cases}
\end{array} \tag{3}
$$

$$u^i(y^i) > u^i(x^i), \text{ for all } i \in C. \tag{4}$$

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[7]The traditional "tragedy of the commons" may be represented by letting each individual be winning with respect to the given resource.

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Inequality (4) is the standard "blocking" condition; members of the coalition C cannot be made better off by recontracting out of the status quo allocation x. The set of conditions given in (4) specify what is feasible for coalition C if C attempts to recontract out. In particular, C can only use those endowments to which some winning subcoalition oc C has a legal claim. That is, for any good $k > \ell_1$, C can only claim γ_k if there is some $d_k \in \mathcal{D}_k$ that is contained in C ($d_k \subseteq C$).

We will say that the economy $\mathcal{E}$ is *socially stable* if its core is nonempty. Finding what, if any, connections exist between income inequality and conditions under which an economy is socially stable is the goal in what follows.

3   The Majority Rule Problem

The economies in section 2 have the capacity for collective ownership of some goods. Depending on the internal allocation procedure for these goods, the presence of collective ownership arrangements may present problems for social stability. The obvious example is with the majority rule arrangement.

Suppose that three individuals collectively hold a single good of which there are γ units. Any majority of the three is *decisive* in allocating the good, meaning that any majority of the three can completely determine the allocation among the three agents. Clearly, the core is empty since any allocation that gives any positive quantity to one agent can be blocked by the other two who can effectively "expropriate" that agent's allocation. The 0-allocation is obviously blocked by the coalition of all three. The instability in this case is completely

analogous to the famous Condorcet "voting paradox."[8]

We observe that this problem does not necessarily go away with the presence of private ownership in some goods. Consider a simple extension of the three-person example to two goods. One good remains collectively held under majority rule while the other is privately held by each of the three people. The property rights structure is displayed in the matrix in Table 1 below.

	good 1	good 2
agents 1	α_{11}	0
agents 2	α_{21}	0
agents 3	α_{31}	0
agents 1,2, and 3	0	γ

Table 1

In Table 1 each of the three agents hold some amount of good 1 privately. Agent i holds α_{i1} units of the good. Also, γ units of good 2 are held collectively and allocated via majority rule. The issue of social stablility is resolved here in the following result.

Claim 1 *Given assumption (A1) on utility functions, for any endowment distribution $(\alpha_{11}, \alpha_{21}, \alpha_{31})$ of good 1, the core of this three-agent example is empty.*

The intuition of this result is that since there are no gains from trade in the first good, the first good cannot be used to prevent the voting cycle in the allocation of the second good

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[8]However, strictly speaking, the majority rule to which we refer here is not actually a majority *voting* game. It refers, rather, to the identification of coalitions that yield effective veto power over any candidate allocation.

(the proof, which is straightforward, is given in the Appendix). Of course, the corollary of this result is that the degree of inequality in private holdings in good 1 has no effect on the social stability of this example. This will be true generally if one considers the degree of inequality in the holdings of any single privately held good. We show in the next section that this will not be true generally if there are many goods in the economy that are privately owned. In that case we show that it matters not only how unequal is the distribution of endowments, but also how specialized the holdings of these endowments are.

4 Specialization and Inequality

4.1 A Leading Example

Consider the following example of an exchange economy. There are three individuals and four goods. Agent i $(i = 1, 2, 3)$, privately holds α_{ik} units of good k. There are γ units of the fourth good.

The fourth good is communally owned among the three agents and is allocated via majority rule. The distribution of endowments and the property rights structure is given in the matrix in Table 1 below.

	good 1	good 2	good 3	good 4
agent 1	α_{11}	α_{12}	α_{13}	0
agent 2	α_{21}	α_{22}	α_{23}	0
agent 3	α_{31}	α_{32}	α_{33}	0
agents 1,2, and 3	0	0	0	γ

Table 2

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From the intuition of the previous example private holdings do not automatically make the economy socially stable in the presence of collectively held goods. The lack of trading opportunities in that example failed to compensate for the cyclic behavior generated by majority rule. Case 1 offers, however, the possibility of such opportunities if the agents are sufficiently *specialized* in their endowment holdings.

We will say an agent i, $(i = 1, 2, 3)$, is *more specialized in good k*, $(k = 1, 2, 3)$, the larger is α_{ik} relative to $\sum_{j \neq i} \alpha_{jk}$. The economy is *more specialized* if every individual is more specialized in some good. By this definition specialization means roughly that each person holds a large amount of some valuable good that cannot be obtained by trade from other individuals in the economy.

This example will be specialized to four polar cases in which the economy is either "specialized" or "not specialized," and individuals' income is either "equal" or "unequal." To make sense of "inequality" it is assumed that the aggregate endowment is constant across goods, and utilities are symmetric across both individuals and across goods (assumptions (A2) and (A3)). If one were to consider a competitive equilibrium in which the privately held goods are traded, assumptions (A2) and (A3) would guarrantee that the realitive prices in equilibrium are unity. In that case it is easy to calculate the income that each individual receives from selling his endowments in the market. Moreover, we can rank incomes of all individuals and make sensible statements about income distributions. Finally, assumption (A4) will be used in some of the cases to show that the core is nonempty.

To further simplify matters, we assume three possible endowment levels of any good k. Either an individual holds β, ζ, or 0 units of a good where $\beta \geq \zeta > 0$. The aggregate

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endowment of each good equals $\beta + \zeta$. We may normalize the price to one so that an individual's income is the "sum of his endowments."

The economy in Table 1 is specialized to the following four afformentioned cases.

4.1.1 The Case of Non-specialization and Unequal Income

Here, the endowments are illustrated in Table 3.

	good 1	good 2	good 3	good 4
agent 1	0	0	0	0
agent 2	ζ	ζ	ζ	0
agent 3	β	β	β	0
agents 1,2, and 3	0	0	0	γ

Table 3

Here, agent 3 is clearly the richest individual. His holdings of all three goods dominate dominate the next richest, agent 2, since $\beta > \zeta$. In a competitive equilibrium in the private goods sector, the prices of all goods are identical. Normalizing the equilibrium price to one gives agent 3 an income of 3β. This economy is not specialized since only agent 3 could be possibly be regarded as sufficiently specialized (if β is large relative to ζ).

Claim 2 *Given assumptions (A1) and (A2) on utility functions and the endowment structure of Table 3, the core of the economy is empty.*

This Claim holds, as in the section 3 example, since the lack of trading opportunities in private holdings makes it impossible for agents who are excluded from a given winning

coalition to compensate members of the winning coalitions in the allocation of the fourth good. Agent 3 clearly has no incentive to trade given his dominant holdings.

4.1.2 The Case of Specialization and Unequal Income

The endowments are now distributed as in Table 4.

	good 1	good 2	good 3	good 4
agent 1	β	0	0	0
agent 2	0	β	ζ	0
agent 3	ζ	ζ	β	0
agents 1,2, and 3	0	0	0	γ

Table 4

This case contrasts sharply with the first. As with case 1, observe here that agent 3 is the richest, having income of $\beta + 2\zeta$. Next in line is agent 2 with income of $\beta + \zeta$. Agent 3 is the poorest with an income of β. However, the economy is characterized by specialization and therefore has the potential for stability.

Claim 3 *Given the endowment distribution of Table 4, for any given ζ and γ, if β is sufficiently large, and agents' utilities satisfy (A1)-(A4), then the core of this economy is nonempty.*

We take the coefficient of variation as a measure of income inequality. Given the pattern in Table 4, under assumptions (A1)-(A4), the coefficient of variation of this income distribution is

$$\sqrt{2}(1 + \frac{\beta}{\zeta})^{-1}.$$

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Notice that given ζ, the coefficient of variation is a decreasing function of β. Thus, for a fixed ζ, a high value of β corresponds to an approximately equal distribution of income. It follows that the core of this economy is nonempty if income inequality is sufficiently small. But keeping fixed ζ and increasing β in this economy not only reduces inequality, it actually increases specialization.

Despite a degree of income inequality in both this and the previous case, stability is acheived in the latter case. The reason is that if some agent holds a large quantity of a unique good which is, relatively speaking, unobtainable without that agent, then that agent is not likely to be excluded from any blocking coalition. But if this is true of each agent, then each agent is less likely to be excluded and so blocking coalitions will less likely form.

4.1.3 The Case of Non-specialization and Equal Income

Here we let $\alpha_{ik} = \delta$ so that each agent is identically endowed with δ units of each of the first three goods as shown in Table 5.

	good 1	good 2	good 3	good 4
agent 1	δ	δ	δ	0
agent 2	δ	δ	δ	0
agent 3	δ	δ	δ	0
agents 1,2, and 3	0	0	0	γ

Table 5

Claim 4 *The core of any economy corresponding to Table 5 is empty.*

This Claim holds, incidentally, without assuming any of the assumptions (A1) to (A4) on utility functions. This case suggests that an arbitrary decrease in income inequality does

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not necessarily increase specialization.

4.1.4 The Case of Specialization and Equal Income

Finally, we consider an economy with both specialization and equality of income as shown in Table 6.

	good 1	good 2	good 3	good 4
agent 1	β	0	ζ	0
agent 2	ζ	β	0	0
agent 3	0	ζ	β	0
agents 1,2, and 3	0	0	0	γ

Table 6

Here each agent's income is given by $\zeta + \beta$. This case demonstrates that the mutual consistency of specialization and income inequality need not be only an asymptotic result. Increasing specialization corresponds to increasing β relative to ζ, since each person is uniquely specialized in exactly one good. We therefore have the following result.

Claim 5 *For any given ζ and γ, and endowments distributed as in Table 6, if β is sufficiently large, and agents' utilities satisfy (A1)-(A4), then the core of this economy is nonempty.*

The four cases above suggest how specialization limits the scope of what we observe with regard to "fair" or "equal" societies.

4.2 A More General Result

We generalize the second case to show that for an arbitrary exchange economy, the more an economy is specialized and the less unequal is income inequality, the more likely is it socially

stable. Furthermore, if each person is sufficiently specialized in exactly one good, then the core is nonempty.

In Table 7 below, the example of section 4 is generalized to n agents. There are ℓ goods with $\ell > n$. Precisely n goods are privately held, while the remaining $\ell - n$ goods are ther collective property of the grand coalition. α_{ik} is agent i's private endowment of good k $(k = 1, \ldots, n)$. We let $\beta = \alpha_{ii}$.

agents\ goods	1	...	...	n	$n+1$	...	ℓ
agent 1	β	α_{12}	...	α_{1n}	0	...	0
$\vdots$	α_{21}	$\ddots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$		$\ddots$	$\vdots$	$\vdots$		$\vdots$
agent n	α_{n1}	...	...	β	0	...	0
agents $1,\ldots,n$	0	...	...	0	γ_{n+1}	...	γ_ℓ

Table 7

The notion "more specialized" in this setting means that β_i is large relative to $\sum_{j\neq i} \alpha_{ji}$. Hence each agent will be assumed to be specialized in the good with the same index.

Proposition *If agents' utilities satisfy (A1)-(A3) then the larger is β holding fixed other endowments, the less unequal is the distribution of income. Furthermore, given any (αjk), $j \neq k$, and $\gamma_{n+1}, \ldots \gamma_\ell$, if β is sufficiently large and agents' utilities also satisfy (A4), then the core is nonempty.*

5 "Political" Specialization and Inequality

Specialization is not specific to private property. Consider the following example with two goods, three people. As before, the second good is allocated via majority rule.

	good 1	good 2
agents 1,2, and 3	β	0
agents 1,2, and 3	0	γ

Table 8

In Table 8 both goods are held collectively. Given the cyclical behavior in allocating good 2, stability seems unlikely. However, this not necessarily the case if some form of "political" specialization holds here. Consider a *unanimity rule* in the allocation of good 1. Unanimity in this context means that each individual is endowed with identical right of veto in the use of the first good. This means that only the coalition of all three agents can recontract (or block) with any amount of the first good. In particular, majority coalitions can only block a candidate allocation by expropriating the entire amount of good 2 but none of good 1. Therefore the following claim is valid.

Claim 6 *Given the endowment distribution in Table 8 and any $\gamma > 0$, if agents' utilities satisfy (A1)-(A4) and β is sufficiently large, then the core of this economy is nonempty.*

In this example inequality cannot be measured in income since both goods are collectively held. There is however an analogous notion measured in "veto power." Each agent in this example wields the same potential right of veto in each good. In the first good such a right

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is realized. In the second, no single agent has that right. In either case rights are symmetric. Each agent has the same political power. In this sense agents are equal. However, as before, equality alone does not guarrantee social stability. If both goods are allocated via majority rule then the core is obviously empty. It is necessary then that each agent actually have some veto power in some good. Each agent is, therefore again, *specialized* in some good.

Consider a final example in which the situation is again described by Table 8. Suppose that instead of unanimity rule, a single agent — agent 1 — is a "dictator" with respect to good 1. That is, he wields a unilateral veto right in good 1. The economy is therefore no longer specialized since only agent 1 has such a right in any good. With this type of inequality, it is straightforward to check that the core is empty. No core allocation can give a positive allocation to agents 2 and 3. However, 2 and 3 will block the allocation that gives everything to agent 1 through their majority veto in good 2.

6 Conclusion

Much of formal theorizing assumes that resources are held privately, that is, by individuals rather than groups. We consider property rights regimes in which some goods are held by all of society. We study the core in order to assess whether these regimes are socially stable. They are socially stable if the core of the economy is nonempty. It turns out that both specialization and the degree of income inequality play a crucial role in permitting social stability of collective property rights regimes.

We have limited ourselves to very particular utility representations which are symmetric across goods and individuals. This serves as a convienient device to make sensible state-

ments about income inequality. To what extent a connection between specialization, income inequality, and social stability remains under more general utility representations is an open question.

7 Appendix

proof of Claim 1 Since there is only one private good, we let α_i denote i's endowment of the first good, α_{i1}. Suppose that allocation $(\bar{\alpha}_i, \gamma_i)_{i=1}^3$ lies in the core. Suppose first that $\gamma_i > 0$ for each i. Since the second good is governed by majority rule, a necessary condition to prevent a coalition, say $\{1,2\}$, from blocking and expropriating all of γ_3, is that $\bar{\alpha}_1 + \bar{\alpha}_2 > \alpha_1 + \alpha_2$. By the same reasoning it follows that $\bar{\alpha}_2 + \bar{\alpha}_3 > \alpha_2 + \alpha_3$ and $\bar{\alpha}_1 + \bar{\alpha}_3 > \alpha_1 + \alpha_3$. Together, the inequalities imply that

$$\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3 > \alpha_1 + \alpha_2 + \alpha_3,$$

which is a contradiction. Suppose now that $\gamma_1 = 0$ for agent 1. Suppose, without loss of generality, that $\gamma_2 > 0$ and $\gamma_3 > 0$. Then, as before, $\bar{\alpha}_1 + \bar{\alpha}_3 > \alpha_1 + \alpha_3$. $\bar{\alpha}_1 + \bar{\alpha}_2 > \alpha_1 + \alpha_2$. Consequently, $\bar{\alpha}_1 > \alpha_1$. This means that agents 2 and 3 can block, split the difference $\bar{\alpha}_1 - \alpha_1$, and be made better off. Finally, suppose that only $\gamma_3 = \gamma$. Consequently, $\bar{\alpha}_3 < \alpha_3$. By strict concavity of agents' utilities it must be possible for 1 to offer some arbitrarily small amount of α_1 to agent 3 in exchange for some small amount ϵ of γ and have both agent 1 and agent 3 be better off. □.

proof of Claim 2 Clearly, any core allocation must give zero to agent 1. Since utilities are strictly concave and identical for both agents 2 and 3 the only possible core allocations must

have each of the two consume their endowments of private holdings. In such a case, either agents 1 and 2 can block by expropriating 3's allocation of good 4 or agents 1 and 3 can block by expropriating 2's allocation of good 4. $\square$.

Balancedness The results that follow depend on standard Theorems of Scarf (1967) and Billera (1973) that give a sufficient condition for a nonempty core of the superadditive NTU game that corresponds to the given economy. To formalize this condition, known as *balancedness*, we first require some simplifying notation. For any coalition C we let $\omega(C) \in \Re^\ell$ denote the aggregate resources available to C. This is defined by the blocking condition (3), letting $\omega(C) = (\omega_k(C))_{k=1,\dots,\ell}$ and

$$\omega_k(C) = \begin{cases} \sum_{i \in C} \alpha_{ik} & \text{iff } k \le \ell_1 \\ \begin{cases} \gamma_k & \text{if there exists some } d_k \in \mathcal{D}_k \text{ with } d_k \subseteq C \\ 0 & \text{otherwise} \end{cases} & \text{if } k > \ell_1 \end{cases} \tag{5}$$

From coalitional aggregate resource constraint ω we can define the *NTU (nontransferable utility) game* $\mathcal{U}$ to be a correspondence $\mathcal{U} : 2^n \longrightarrow\longrightarrow \Re^n$ defined by:

$$\mathcal{U}(C) \equiv \left\{ v \in \Re^n \,\middle|\, \exists (x^i)_{i \in C} \text{ with } \sum_{i \in C} x^i \le \omega(C), \text{ s.t. } v^i \le u^i(x^i), \ \forall i \in C \right\} \tag{6}$$

for each coalition C. The set $\mathcal{U}(C)$ is simply the set of utility vectors that are no better for the individuals than the some feasible utility vector.

We now define the notion of balancedness and state the Theorem that guarrantees that the core of an economy is nonempty.

A collection $\mathcal{B}$ of subsets $B \in I$ is *balanced* if there is a collection of weights $(\lambda_B)_{B \in \mathcal{B}}$ (called "balancing weights") where $\lambda_B \ge 0$, each B, such that for each individual i,

$$\sum_{\{B \in \mathcal{B} \mid i \in B\}} \lambda_B = 1.$$

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An NTU game is *balanced[9]* if, for any balanced collection $\mathcal{B}$ of I

$$\bigcup_{B \in \mathcal{B}} \mathcal{U}(B) \subseteq \mathcal{U}(I) \tag{7}$$

Scarf's Theorem therefore states that any economy that gives rise to a balanced NTU game has a nonempty core.

proof of nonemptiness of the core for Claim 3, Claim 5 and the proposition To show that the core of an economy $\mathcal{E}$ is nonempty under assumptions (A1)-(A4), it suffices to show that the corresponding NTU game $\mathcal{U}$ defined by (6) satisfies (7) for any balanced collection $\mathcal{B}$. Under assumption (A1) Billera (1974) has shown that (7) is satisfied iff for any allocation (x^i) that satisfies $\sum_{i \in B} x^i \le \omega(B)$ for each $B \in \mathcal{B}$,

$$\sum_{B \in \mathcal{B}} \lambda_B (u^i(x^i))_{i \in B}, \overbrace{0, \ldots, 0}^{n - |B|} \tag{8}$$

where (λ_B), are the balancing weights and recall that $n = |I|$. Hence, it suffices to show (8) for any balanced collection $\mathcal{B}$. By assumptions (A2) and (A3), (8) holds if

$$\sum_{B \in \mathcal{B}} \lambda_B \bar{u}\left(\frac{\omega(B)}{|B|}\right) \le \bar{u}\left(\frac{\omega(I)}{n}\right) \tag{9}$$

where $\frac{\omega(B)}{|B|}$ is the coalitional average allocation; it is the aggregate resource vector $\omega(B)$ divided equally among $|B|$ agents. (Also, recall that $\bar{u} = (\overbrace{u, \ldots, u}^{n})$.)

We will show that (9) will hold for the economy in Table 7 (and, therefore, the special cases of Tables 4 and 6) if β is sufficiently large. By the construction of the economy in

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[9]Sometimes this condition is referred to as *quasi-balanced* since this condition is sufficient for proving a nonempty core when agents have quasi-concave utility functions. Billera's stronger notion is often called *balanced*. When agents' utilities are concave the two coincide.

Table 7, for each $k = 1, \ldots, \ell$, and each $B \subseteq I$,

$$\omega_k(B) = \begin{cases} \sum_{\{j \in B \mid j \neq k\}} \alpha_{jk} & if \quad k \notin B \\ \sum_{\{j \in B \mid j \neq k\}} \alpha_{jk} + \beta & if \quad k \in B \\ 0 \text{ or } \gamma_k & if \quad k > n \end{cases} \tag{10}$$

In particular,

$$\omega_k(I) = \begin{cases} \sum_{j \neq k} \alpha_{jk} + \beta & if \quad k = 1, \ldots, n \\ \gamma_k & if \quad k > n \end{cases} \tag{11}$$

Without loss of generality let all $|B|$ be sufficiently large so that $\omega_k(B) = \gamma_k$ if $k > n$.

By (9), (10), and (11), it suffices to show that

$$\lim_{\beta \to \infty} \left[\bar{u} \left(\frac{\sum_{j \neq 1} \alpha_{j1} + \beta}{n}, \ldots, \frac{\sum_{j \neq n} \alpha_{jn} + \beta}{n}, \frac{\gamma_{n+1}}{n}, \ldots, \frac{\gamma_\ell}{n} \right) \right.$$

$$\left. - \sum_{B \in \mathcal{B}} \lambda_B \bar{u} \left(\overbrace{\ldots, \frac{\sum_{\{j \in B \mid j \neq k\}} \alpha_{jk}}{|B|}, \ldots,}^{n - |B|}, \overbrace{\ldots, \frac{\sum_{\{j \in B \mid j \neq k\}} \alpha_{jk} + \beta}{|B|}, \ldots,}^{|B|} \overbrace{\frac{\gamma_{n+1}}{|B|}, \ldots, \frac{\gamma_\ell}{|B|}}^{\ell - n} \right) \right]$$

$$> 0 \tag{12}$$

Observe that if $\sum_{j \neq k} \alpha_{jk} = 0$, then (12) follows immediately from the curvature assumption (A4).

Let $\hat{\beta}$ satisfy the inequality in (12) assuming that $\sum_{j \neq k} \alpha_{jk} = 0$. That is, $\hat{\beta}$ satisfies

$$\bar{u}(\frac{\hat{\beta}}{n}, \ldots, \frac{\hat{\beta}}{n}, \frac{\gamma_{n+1}}{n}, \ldots, \frac{\gamma_\ell}{n}) - \sum_B \lambda_B \bar{u}(\overbrace{0, \ldots, 0}^{n - |B|}, \overbrace{\frac{\hat{\beta}}{|B|}, \ldots, \frac{\hat{\beta}}{|B|}}^{|B|}, \frac{\gamma_{n+1}}{|B|}, \ldots, \frac{\gamma_\ell}{|B|}) > 0. \tag{13}$$

However, by the strict concavity and monotonicity of the symmetric utility function, (13) implies that the inequality in (12) also holds for the given value $\hat{\beta}$. $\qquad \square$

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