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On Positional Games With Perfect Information and Their Applications

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The game theory was firstly used for description of economic phenomena and social interaction. But there are certain type of perfect information games (PI-games), the so-called positional game or Banach-Mazur games, which so far have not been applied in economy. The perfect information positional game is defined as the game during which at any time the choice is made by one of the players who is acquainted with the previous decision of his opponent. The game is run on a sequential basis. The aim of this paper is to discuss selected Banach-Mazur games and to present some applications of positional game. This paper also shows new theoretical example of a determined PI-game, based by theoretical overview. All considerations are pure theoretical and based by logical deduction.

Keywords: Banach-Mazur games, a winning strategy, finite and infinite games of perfect information, the determinacy of PI-game, axiom of choice, rule of auction, Dutch auction, chess

Introduction

One of the most prominent games of perfect information (called PI-games) is a two-player strategy board game played on a checkered game board, otherwise known as chess. For many years the principles of chess were unintentionally laying foundations for the development of the latest software. Perfect information refers to the fact that at each time only one of the players moves. The game depends exclusively upon their unrestricted choices, they remember the past decisions, and in principle they know all possible futures of the game. The first published paper devoted to general infinite PI-games is due to Gale and Stewart (1953), but the first interesting theoretical infinite PI-game was invented by Mazur about 1935 in the Scottish Book (The Scottish Book, 1941; Mauldin, 1981). Positional games were created in 1940's by a remarkable range of Polish mathematicians, belonging to the Lwow School of Mathematics. Owing to the authors' names they are otherwise known as Banach-Mazur games.

Research Subject and Design

Research Subject

The subjects for the present study are PI-games. This paper aims to address the most common versions of Banach-Mazur games, their modifications, and their possible applications.

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Research Framework

This study will establish the theoretical framework by the logical deduction, based on certain types of PI-games otherwise known as Banach-Mazur games. This paper demonstrates some definitions and theorems, which are connected with those games. It also attempts to present some applications of the PI-games and illustrates a new theoretical example based on Steinhaus' puzzle.

The Banach-Mazur Games and Their Applications

The relevant issue in the area of competitiveness is the game displaying an infinite number of strategies. The overwhelming majority of dilemmas related to the above games were defined in the period ranging from 1935 to 1941 and incorporated into the so-called Scottish Book. The Scottish Book referred to a notebook purchased by a wife of Stefan Banach and used by mathematicians of the Lwow School of Mathematics (such as Stanisław Mazur, Stanisław Ulam, and Hugo Steinhaus) for jotting down mathematical problems meant to be solved. The Scottish Book used to be applied for almost six years. Many problems presented therein were created in previous years and not all of them were solved. After the World War II, Łucja Banach brought the Book to Wrocław, where it was handwritten by Hugo Steinhaus and sent in 1956 to Los Alamos (USA) to Stanisław Ulam. Ulam translated it into English, copied at his own expense and dispatched to a variety of universities. The book in question proved to enjoy such a great popularity that it was soon published and edited—mainly in English (Mauldin, 1981). The Scottish Book presents the following game No. 43 elaborated by Stanisław Mazur (The Scottish Book, 1941).

Example 1. (Mazur)

Given is a set E of real numbers. A game between two players I and II is defined as follows: Player I selects an arbitrary interval d_1, player II then selects an arbitrary segment (interval) d_2 contained in d_1; then player I in turn selects an arbitrary segment d_3 contained in d_2, and so on. Player I wins if the intersection d_1, d_2, ..., d_n,... contains a point of set E; otherwise he loses. If E is complement of a set of the first category, there exists a method through which player I can win; if E is a set of the first category, there exists a method through which player II will win.

Problem. It is true that there exists a method of winning for player I only for those sets E whose complement is, in certain interval, of first category within a certain interval; similarly, does a method of win exist for player II if E is a set of first category (Kuratowski & Mostowski, 1978)?

Addendum: Mazur's conjecture is true.

Modifications of Mazur's game are as follows.

Example 2. (Ulam)

There is given a set E of real numbers. Both players select in turn one of the digits: 0 or 1. Player I wins if the number formed by those digits in a given order (in the binary system) belongs to E. Which E will allow player I (or player II) to win?

Example 3. (Banach)

There is given a set of real numbers E. The two players I and II in turn give real numbers which are positive and such that a player always gives a number smaller than the last one given. Player I wins if the sum of the given series of numbers is an element of the set E. The same question is as for example 2.

Example 4. (Some Popular Modification of Banach-Mazur Game)

Two players choose alternatively one digit from the set 0, 1, …, 9. Their choices generate an infinite sequence of digits, e.g., 5791… Such a sequence may be denoted by the number 0.5791…∈[0, 1]. Before the game begins, a subset X of the section [0, 1] is to be defined. Player I will win provided that the mutually generated number belongs to the set concerned. Player II wins if the number at issue does not fall within the set in question.

The conclusion seems inescapable that the above game has a winning strategy. One may assume that at the beginning the players should establish the set X taking the following form [0.1, 0.3]. Having arranged such a set, player I may initially select the digit 1 or 2, which strategy makes him win the game automatically. The selection of any other digit will result in the win of player II.

Formally, the PI-games may be described as follows:

Let A denote the set of strategies of player I, B—the set of strategies attributable to player II.

$\varphi: A \times B \to \overline{\Re}$, where $\overline{\Re} = \Re \cup \{-\infty, +\infty\}$ ($\Re$ is the set of real numbers).

This game is played as follows:

Player I chooses $a \in A$ and player II chooses $b \in B$. Both chooses are made independently and without any knowledge about the choice of the other player. Then player II pays to I value $\varphi(a,b)$. $\varphi(a,b) < 0$ means that II gets from I the value $|\varphi(a,b)|$.

Idea of an infinite game of perfect information is the following:

let $\omega \in \{0, 1, 2...\}$,

there is a set P called the set of choices,

player I chooses $p_0 \in P$, next player II chooses $p_1 \in P$, than I chooses $p_2 \in P$, etc..

There is a function $f : P^{\omega} \to \overline{\Re}$, such that the end player II pays to I the value $f(p_0, p_1 ...)$.

Definition 1. The triple $\langle A, B, \varphi \rangle$ is said to be a game of perfect information (PI-game) if there exists a set P such that A is set of all functions.

$$A = \left\{ a : \bigcup_{n<\omega} P^n \to P \right\}, \text{ where } P^0 = \{\phi\},$$

$$B = \left\{ b : \bigcup_{0<n<\omega} P^n \to P \right\},$$

and there exists a function $f : P^{\omega} \to \overline{\Re}$ such that $\varphi(a,b) = f(p_0, p_1 ...)$,

where:

$p_0 = a(\phi), p_1 = b(p_0), p_2 = a(p_1), p_3 = b(p_0, p_2), p_4 = a(p_1, p_3)$ … (see Figure 1).

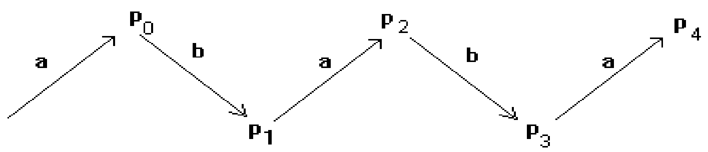

Figure 1. PI-game.

A game $\langle A, B, \varphi \rangle$ defined in this way will be denoted $\langle P^{\omega}, f \rangle$ or $\langle P^{\omega}, X \rangle$.

The sequence $p = (p_0, p_1 ...)$ is called a game, any finite sequence $q = (p_0, ..., p_{n-1}) \in P^n$ is called position.

f is a characteristic function of a set $X \subseteq P^{\omega}$,

$$\begin{cases} f(p) = 0 & \text{if} \quad p \notin X \\ f(p) = 1 & \text{if} \quad p \in X \end{cases}$$

The player I wins the game if $f(p) = 1$ and II wins the game if $f(p) = 0$.

Definition 2 (Mycielski, 1992). A game $\langle A, B, \varphi \rangle$ is called determined if:

$$\inf_{b \in B} \sup_{a \in A} \varphi(a,b) = v = \sup_{a \in A} \inf_{b \in B} \varphi(a,b) \tag{1}$$

where v is value of the game (common value v of both sides of this equation is called the value of the game $\langle A, B, \varphi \rangle$)

Remark: *A* game is determined if and only if the game has a value.

A game is not determined if:

$$\inf_{b \in B} \sup_{a \in A} \varphi(a,b) < v < \sup_{a \in A} \inf_{b \in B} \varphi(a,b) \tag{2}$$

Note: If the game is not determined, then the left-hand side of (1) is larger than the right-hand side of (1).

If the game has a value v and there exists an a_0 such that $\varphi(a_0, b) \geq v$ for all b, then a_0 is called an optimal strategy for player I. If $\varphi(a, b_0) \leq v$ for all a, the b_0 is called an optimal strategy for player II.

$\langle P^{\omega}, f \rangle$ may be defined as a win for I or a win for the II if $\langle P^{\omega}, f \rangle$ has value 1 or 0, respectively. If $f : P^{\omega} \to \overline{\Re}$ has the property that there exists an n such that $f(p_0, p_1 \cdots)$ does not depend on the choice p_i with $i > n$, then $\langle P^{\omega}, f \rangle$ is called a finite game.

The following theorems are true:

Theorem 1 (Mycielski, 1992): Every finite game has a value.

Proof ((Mycielski, 1992), proposition 2.1, p. 45).

Theorem 2 (Mycielski, 1992): There exist sets $X \subseteq \{0, 1\}^{\omega}$ such that game $< \{0, 1\}^{\omega}, X >$ is not determined.

Proof ((Mycielski, 1992), proposition 3.1, p. 46).

Theorem 3 (Mycielski, 1992): If the set $X \subseteq P^{\omega}$ jest closed or open, then the game $\langle P^{\omega}, X \rangle$ is determined.

Proof ((Mycielski, 1992), proposition 3.2, p. 46).

Theorem 4 (Mycielski, 1992): If player II has a winning strategy in Banach-Mazur game*,* then X is not countable.

Another interpretation of Banach-Mazur games.

Example 5. (Mycielski, 1992)

A set S is given. Player I splits S into two parts. Player II chooses one of them. Again, Player I splits the chosen part into two disjoint parts and II chooses one of them, etc.. Player I wins if and only if intersection the chosen parts is not empty and player II wins if and only if it is empty.

Remark: Player I has a winning strategy if and only if $|S| \leq 2^{\aleph_0}$, and player II has a winning strategy if $|S| \leq \aleph_0$, where $|S|$ means cardinality of set S, $\aleph_0$ is alef zero—cardinality of integer numbers.

Theorem 5 (Mycielski, 1992): If player II has a winning strategy for Banach-Mazur game, then $|S| \leq \aleph_0$.

The proofs of above theorems have used the Axiom of Choice (Mycielski, 1992).

Mycielski and Steinhaus conjecture that the Axiom of Choice is essential in any proof of the existence of sets $X \subseteq \{0, 1\}^{\omega}$ such that the game $< \{0, 1\}^{\omega}, X >$ is not determined. In the same order of ideas, theorem 5 shows that Continuum Hypothesis ($2^{\aleph_0} = c$ - continuum or there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$) is equivalent to the determinacy of natural class of PI games.

Another Example Banach-Mazur Game

While creating the original variants of Banach-Mazur games, one may apply the properties of finish sets. That ensures that the game in question may be deemed as determined.

Example 6.

Two players select by turns certain numbers from the interval [0, 1], while following a given pattern: player 1 picks out number x_1 belonging to the interval [0, 1]. Subsequently, player 2 chooses number x_2, keeping in mind that x_1 should be contained in the first half and x_2 in the second half of the interval at hand. Consequently, player 1 selects x_3 in such a manner that x_1, x_2, x_3 each belongs separately to one of the equal sections of the interval concerned. As a result, player 2 gives number x_4 in such a way that each of the numbers x_1, x_2, x_3, x_4 falls into the scope of distinct quarters comprising the interval in question. The procedure may be continued without limitation.

The game is won by the player who as the last one selects the number fulfilling the game conditions, i.e., for the n^{th}-time he gives number x_n, where each number establishing the sequence x_1, x_2, x_3, …, x_n belongs to one of separate parts of the interval [0, 1] divided into n equal sections.

Commentary to the Solution

The game is based upon one of the tasks presented by Steinhaus (1964) in his book *One hundred problems in elementary mathematics*. One may prove that there exists a sequence of numbers which adheres to the game requirements, hence the game may be deemed as completed. When applying theorem 1, the game may be found determined, i.e., one player may adopt a winning strategy. It is common knowledge that for n = 10 one may suggest several sequences of numbers x_1, x_2, x_3, …, x_{10} fulfilling the game conditions. Below there are two examples:

(a) 0.95 0.05 0.34 0.74 0.58 0.17 0.45 0.87 0.26 0.66

(b) 0.06 0.55 0.77 0.39 0.96 0.28 0.64 0.13 0.88 0.48

Numbers from the first sequence may fall into the separate sections of the interval as presented in Table 1.

Table 1

Numbers of the Parts Into Which the Interval [0, 1] Was Divided

Parts numbers	2	3	4	5	6	7	8	9	10
0.95	2	3	4	5	6	7	8	9	10
0.05	1	1	1	1	1	1	1	1	1
0.34		2	2	2	3	3	3	4	4
0.74			3	4	5	6	6	7	8
0.58				3	4	5	5	6	6
0.17					2	2	2	2	2
0.45						4	4	5	5
0.87							7	8	9
0.26								3	3
0.66									7

The first row lists the numbers of sections into which the interval has been divided. The first column contains the sequence of numbers $x_1, x_2, x_3, \ldots, x_{10}$. The intersection of a row and column indicates the interval part into which a given number falls upon a specific division. None of the columns contains two identical numbers.

In the event of the second sequence the value of n may reach 14 ($n = 14$) through expanding the list by the following numbers: 0.19; 0.71; 0.35; 0.82, i.e.,

0.06 0.55 0.77 0.39 0.96 0.28 0.64 0.13 0.88 0.48 0.19 0.71 0.35 0.82

Since the numbers 0.35 and 0.39 are contained between $5/15 \approx 0.33$ and $6/15 \approx 0.4$, the above example may not be supplemented by the 15th number while obeying the game rules. In the instant case player II is the winner, therefore the game may be considered finished.

In the case of other sequences, the game may be found finished even upon the significantly lower number of movements.

Several Remarks on PI Games Applications

Banach-Mazur games used to enjoy great popularity, mainly among mathematicians. The analysis of those games revolved around one chief question: Is there a winning strategy guaranteed for any of the players? Taking into account the Axiom of Choice, already at the beginning of the 20th century it was proven that there were certain sets X for which neither player may adopt a winning strategy. The introduction of a new axiom to a set theory, known as the axiom of determination, significantly facilitated the search for a winning strategy. Different variants of Banach-Mazur games were analyzed in terms of the satisfaction of determination condition. There is a presumption that there exists a set whose subsets are assigned a non-trivial measure which is a countably additive extension, vanishing on points and taking the value 0 or 1. All the subsets in the set concerned are determined, and at least one of them has a winning strategy.

Banach-Mazur games can be classified as infinite multi-stage games with perfect information. In practice, they are illustrated by the situations where the winner takes everything (compare the Colonel Blotto Game). Moreover, the games where the win is already determined at the initial stage, rely on a first come, first served basis. In terms of economy, such a game corresponds to the auction where a product (item) is offered up for bid. In such a case the buyer who wins the auction takes everything. Analogically to many positional games, the first participant submitting a bid determines the course of auction. Whenever the bid does not reach the sale price offered by the seller, other bidders may outbid the reserve price or withdraw from the auction. For instance, the digit selected by the participant initiating the game may not guarantee that the number generated in a following sequence will belong to a given interval (compare example 4). Notwithstanding the type of auction the optimal strategy adopted by a bidder resides in offering such a price which will warrant the win (i.e., the purchase of a product), however, which does not exceed his own valuations of an item in question. In the event of Dutch auction the price is gradually lowered until some auctioneer is willing to accept the announced price—such a participant wins the auction. It is a typical example of a game based on a first come, first served ground. The games introduced in previous examples serve as an illustration for the Dutch auction.

The most common, "finite" positional game with perfect information is chess, which laid foundations for artificial intelligence algorithms applied in various domains, including the construction of dynamic equilibrium models as well as the description of economic systems lacking the equilibrium. In 1949 the American mathematician C. E. Shannon set up the guidelines for computer chess game, which were being gradually improved in ensuring years. In the 1950s the lion's share of the artificial intelligence research focused predominantly upon the chess game basis, as they were considered a good model for human intelligence. From the historical angle, what may be perceived as a breakthrough point is the match held in May 1997 in the Manhattan district, where the chess champion Garry Kasparov was defeated by the IBM's computer Deep Blue. Up to that moment the chess was deemed as one of several games in which a human being could prevail over the machine. The reason for that phenomenon may be explained by the fact that the number of variants applicable to one game composed of 100 movements amounts 10^{155}. The computers of older generation used to calculate every operation and thus were not able to analyze all possible options within three permissible minutes. Conversely, the players aimed to select the best variants, as they were not capable of computing possibilities. The pivotal role was played both by their knowledge and experience. Notwithstanding the significant advancement of technology which facilitated the computerized data processing, the useless strategies were removed from the available algorithms and 600,000 chess openings as well as a considerable set of chess masters' games were imprinted to the machine languages.

Unlike the previous algorithms, the newly created methods prompted the computer to search for the move usually made by the top players. In 1996 Kasparov won the match in Philadelphia with a computer, where the score was 4:2. The match initiated a real battle against the human mind, which resulted in the further enhancement of computer's strategies upon modifications reflecting the thinking process conducted by a chess champion while attempting to predict the consecutive moves of his opponent. One may claim that in 1997 Kasparov was almost forced to play a game not only with a technologically modified computer, but also with "the spirit of his predecessors".

In 2011 IBM developed a smart computer named Watson, which understands questions posed in natural language and is able to gather as well as browse an enormous amount of information more effectively than a human being. Having competed against two masters of American show Jeopardy, Watson received the first prize. It acquires a massive amount of data extracted from medical periodicals and rapidly analyses thousands of particular medical cases, which skill is unattainable even by the most talented doctors. Watson presents the best options which lay foundations for a further diagnosis. The works aiming to develop computers of new generation, i.e., quantum computers which can employ a specific class of quantum phenomena and make independent decisions, are still underway.

The artificial intelligence is more and more often applied in energetics—to create systems not only monitoring the course of specific processes, but also involved in planning and decision-making procedures. It is used also for the purposes of image processing, e.g., in cameras, supporting financial decisions as well as in many other domains of everyday life.

It is worth mentioning one of the most fascinating personalities of sports and science, Robert James Fischer who was famous for his exceptionally talented and rebellious mind. He played hundreds of outstanding games, implemented many innovative solutions and introduced the so-called Fischer clock enabling to keep track of the total time each player takes for his or her own moves. Due to some personal reasons he was not able to play the game with the computer. Just wonder who would have won in such a competition.

In the light of the game theory, it should be emphasized that due to its limited range of strategies the chess game is indeterminate, of which fact the vast majority of chess players remain unaware.

Conclusions

On the basis of the PI-games, specific rules of auction may be formulated. The games concerned can serve as an exemplification of certain situation where the winner takes everything. A “finite” PI-game with perfect information, i.e., chess can be used for artificial intelligence algorithms.

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