Non-linear Prices and the Optimal Allocation of Public Goods*

by

Brian R. Binger

and

Elizabeth Hoffman

Purdue University
March, 1983

Draft prepared for presentation at the Public Choice Society Meetings, March 24-26, 1983, Savannah, Georgia

Comments are invited

*The authors wish to thank Paul Beaumont, Bob Forsythe, James Foster, Tom Hammond, John Ledyard, Jim Moore, John Pomery, and Paul Thomas for their generous help and criticism. All errors are, of course, our own.

I. Introduction

In the last decade there has been considerable interest in decentralized procedures for both choosing the scale of and financing public goods projects. The literature has focused on two different types of decentralized allocation schemes: demand-revealing mechanisms (e.g. Green and Laffont, 1977; Groves and Ledyard, 1977) and dynamic tatonnement planning procedures (Dreze and de la Valle Poussin, 1971; Malinvaud, 1971; Mas-Colell, 1980). In addition, despite the free rider problem, voluntary subscription has received some renewed attention (Brubaker, 1975). Kalai (1980) has proposed a voluntary contractual procedure which induces cooperative behavior.

II. Auxiliary Market Mechanisms for Allocating Public Goods

This paper outlines a new method of choosing the optimal quantity of a public good and financing it. The allocation mechanism combines some of the features of demand revealing mechanisms, tatonnement planning procedures and voluntary subscription drives. It is a dynamic tatonnement process which resembles a Walrasian auction. It has the property that, at each iteration of the process, participants reveal their true demands for the public good when, they play Cournot-Nash utility-maximizing strategies. Two aspects of our mechanism are new: 1) each participant signals his total demand for the public good as the sum of two different messages; and 2) the mechanism mimics an auxiliary market for "public goods tickets." In keeping with tatonnement procedures in general, no transactions take place out of equilibrium. In essence, the outcome is a contract which internalizes the external benefits of the public good.

To begin the mechanism, the government or builder of a public good offers to sell commitments to build units of. the public good at marginal cost. In what we call the ticket market, each participant is given a personalized ticket price and asked to state his demand for units at that price. In what we call the subsidized subscription market, each participant is told that if he subscribes to units of the public good at marginal cost, he will receive a given per unit subsidy from the central authority. The subsidy is the sum of the personalized ticket prices paid by the other participants. A public goods equilibrium is defined as a quantity of the public good and a set of personalized prices, which has the property that, given those personalized prices, each individual demands the equilibrium quantity of the public good, defined as the sum of his ticket and subsidized purchases.

The operation of the mechanism can be viewed as a market with special equilibrium conditions. Purchases in the ticket market signal demands to match units purchased in the subsidized market. Person i's ticket price is his contribution to the subsidy paid to j. Purchases in the subsidized subscription market signal supplies to match units purchased in the ticket market. At an equilibrium in the ticket market, demands by i in the ticket market would equal supplies by all $j \neq i$ from the subsidized subscription market. It is as though each subsidized subscription unit came with a set of personalized tickets, one for each other potential user of the public good. The tickets would provide the assurance to members of society receiving them that the commitment to build each unit has actually been contracted for. Individuals wishing to have units of a public good built would purchase commitments to build and simultaneously sell to the other potential users of the public good the personalized tickets received with the commitment.

Suppose, for example, that if the marginal cost of the public good were $1, an economy composed of three individuals would demand 15 units, with Lindahl equilibrium marginal valuations of 3/15 for person 1, 5/15 for person

2, and 7/15 for person 3. The following ticket market equilibrium would achieve the Lindahl equilibrium.

Person	Tickets Purchased	Units Subscribed to
1	6	9
2	10	5
3	14	1

Thus, person 1 demands 6 tickets and purchases 9 subsidized subscription units. Along with the 9 subscription units come 9 tickets to be supplied to person 2 and 9 to be supplied to person 3. Person 1's demand for 6 tickets equals the aggregate supply of tickets (5 from person 2 and 1 from person 3). In fact, 2's demand (10) equals the aggregate supply from 1 and 3 (9+1) and 3's demand (14) equals the aggregate supply from 1 and 2 (9+5). Finally, tickets plus subscription units equals 15, the Lindahl equilibrium, for each person.

We employ a 2-good (one public, one private) model with constant marginal cost. The model can either be thought of as a very simple general equilibrium model with a linear transformation function or as a "small community" partial equilibrium model. We outline two mechanisms for organizing a ticket market and pricing subscription tickets. The first, a linear pricing scheme, is similar to a Walrasian tatonnement process. An auctioneer, knowing the marginal cost, presents each participant with a constant unit price for his personalized tickets, which he buys from the other participants. The participant responds by indicating how many units of the public good he will subscribe to himself and the total number of personalized tickets he will buy from the other individuals (thereby subsidizing their subscriptions), given the marginal cost, his personalized ticket price and the demands of the other participants for their personalized tickets (which he obtains when he subscribes to commitments to build units of the public good).

If all participants behave competitively (i.e., are price takers in their personalized ticket prices), the outcome of this process is a Lindahl equilibrium and is equivalent to a unanimous agreement among consumers about the size of the public goods facility and the donation each consumer will make. Each individual's personalized ticket price is his Lindahl price and the equilibrium allocation satisfies the Lindahl-Samuelson conditions (i.e., the sum of the marginal willingnesses-to-pay equals the marginal cost of providing that size facility). 'However, since each individual is the only buyer of his own personalized tickets there is little reason to believe he would behave competitively. It is simply too easy to manipulate such a market mechanism by playing non-Nash strategies. Thus, the monopsony or thin market problem, which Arrow (1969) discussed in connection with auxiliary markets, remains. Moreover, there is no unique ticket-subscription unit equilibrium. To see this, consider the example described above. The following ticket allocation is also an equilibrium.

Person	Tickets Purchased	Units Subscribed to
1	10	5
2	10	5
3	10	5

In fact, because the personalized ticket price and the subsidized subscription price are the same in equilibrium, there will be an infinity of such equilibria. The total quantity of the public good will be uniquely determined, but individuals will be indifferent between any two combinations of ticket and subscription purchases which sum to the same total quantity.

The second mechanism is a non-linear pricing scheme. To begin this process each participant is given a non-constant price function for his personalized tickets. If a participant buys personalized tickets, he pays the government the integral of his personalized ticket price function up to the total number of personalized tickets he buys. The government agrees to pay each seller of personalized tickets to a particular buyer the product cf the number of personalized tickets that sellers sells to that buyer and that buyer's marginal valuation along his personalized ticket price function, given all the personalized tickets he buys from all sellers. Given their personalized ticket price functions and the stated demands and marginal valuations of the other participants, participants simultaneously indicate how many personalized tickets they will buy and how many units of the public good they will subscribe to. The budget is balanced by charging each person times the sum of the deficits generated by the other n-1 participants from their purchases along their non-linear pricing schedules.

Such a scheme has important advantages over other pricing schemes. First, it eliminates the monopsony problem of the linear pricing scheme by making the prices paid for inframarginal personalized tickets unaffected by the total number of personalized tickets purchased. Such a personalized ticket price function resembles the supply function facing a discriminating monopsonist. Second, like the Groves-ledyard (1977) mechanism, the budget is balanced by charging each participant a lump sum tax which is not dependent on his own decisions. Thus, also, like the Groves-Ledyard mechanism, the Nash equilibrium is both incentive compatible and Pareto optimal and it satisfies the Lindahl-Samuelson conditions. Third, it has a very desirable new property which we illustrate with two examples. For at least some classes of utility functions, there exists a unique equilibrium non-linear pricing function which applies to all participants and supports a unique equilibrium allocation. With Cobb-Douglas utility functions we show that a unique upward-sloping "linear" non-linear price function defines an equilibrium. We conjecture that such a unique pricing function exists in general for neo-classical utility functions. Thus, if any adjustment of the pricing function is necessary to achieve an equilibrium only one parameter need be adjusted for all participants. The mechanism is therefore very simple to implement. In addition, if all participants face the same non-linear pricing function, each participant has less incentive to try to manipulate the mechanism by trying non-Nash responses. We conjecture that the incentive to manipulate is similar to the incentive exhibited by the competitive mechanism (Hurwicz, 1972). If that is the case, the incentive should become small as the number of participants gets large (Roberts and Postlewaite, 1976).

To see that this mechanism yields a unique equilibrium consider the original example on page 6 with non-linear price schedule 1/30. Marginal valuations of 3/15, 5/15, and 7/15, translate into 6/30, 10/30, and 14/30. Figure 1 illustrates that person 1 demands exactly 6 tickets, person 2 demands exactly 10 tickets, and person 3 demands exactly 14 tickets.

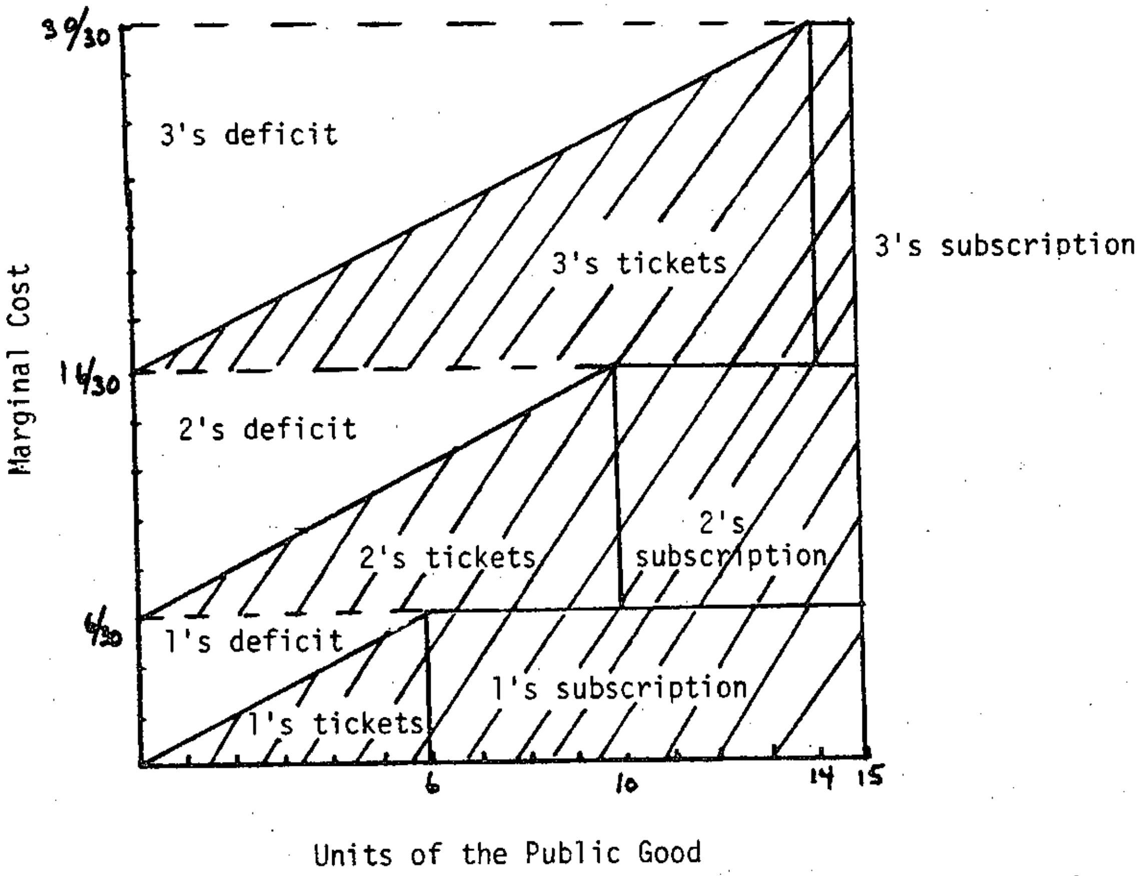
After outlining the linear and non-linear pricing schemes, we show that the consumer's maximization problem under the non-linear pricing scheme satisfies the second order conditions for constrained utility maximization. We also outline adjustment procedures for the subsidies and the unique non-linear pricing function which bring computer simulated markets to equilibrium.

Turning now to our model, we begin our formal presentation with the linear pricing scheme.

1

Figure 1

Purchases of Tickets and Subscriptions
with Unique Non-linear Price Function of 1/30



III. The Model

A. Linear Pricing Scheme

$i = 1, \ldots, n$ represents the consumers

w_i = consumer i's personalized price for a ticket he buys from another participant

v_i = consumer's "true" marginal valuation for tickets

y_i = units of the public good which i subscribes to. Note that y_i also equals the personalized tickets which i receives for each $j \neq i$

$Y \equiv \sum_{i=1}^{n} y_i$ = the size of the public goods facility

t_i = personalized tickets bought by i

p_y = marginal cost of Y

x_i = i's consumption of the private good

p_x = price of the private good

$U^i(Y, x_i)$ = i's neo-classical utility function

M_i = i's money income

By requiring that no trades take place out of equilibrium, we impose the following equilibrium condition as a constraint:

$$t_i = \sum_{j \neq i} y_j \qquad i = 1, \ldots, n$$

This implies that each participant must buy y_j tickets from participant j before the public good can be built. Another way of stating the equilibrium condition is to say that i must sell y_i tickets to each j at j's ticket price, w_j. Thus, at equilibrium, i receives as subsidy:

$$y_i \sum_{j \neq i} w_j$$

from selling tickets to all $j \neq i$. This ensures that all participants purchase the same units of the public good as combinations of ticket purchases and direct subscriptions.

Now, if we assume each consumer takes his personalized ticket price as given, i's decision problem is to maximize utility by purchasing private goods and personalized public goods tickets and by subscribing to units of the public good, subject to his budget constraint:

$$\text{Max } U^i(Y, x_i) = U^i(\sum_{i=1}^{n} y_i,\ x_i)$$

$$\text{subject to: } M_i = p_y y_i + p_x x_i - y_i \sum_{j \neq i} w_j + w_i t_i$$

Note: $p_y y_i$ = i's expenditures on subscriptions to the public good

$y_i \sum_{j \neq i} w_j$ = i's receipts from selling personalized tickets to $j \neq i$

$w_i t_i$ = i's expenditures on personalized tickets.

Substituting t_i for $\sum_{j \neq i} y_j$, can write the Lagrangian:

$$L_i = u_i(t_i + y_i,\ x_i) + \lambda_i\ (M_i - p_y y_i - p_x x_i - w_i t_i + y_i \sum_{j \neq i} w_j) \qquad (1)$$

The first order conditions are:

$$\frac{\partial L_i}{\partial y_i} = U_y^i - \lambda_i(p_y - \sum_{j \neq i} w_j) = 0 \qquad (2)$$

$$\frac{\partial L_i}{\partial t_i} = U_y^i - \lambda_i w_i = 0 \qquad (3)$$

$$\frac{\partial L_i}{\partial x_i} = U_x^i - \lambda_i p_x = 0 \qquad (4)$$

$$\frac{\partial L_i}{\partial \lambda_i} = M_i - p_y y_i - p_x x_i - w_i t_i + y_i \sum_{j \neq i} w_j = 0 \qquad (5)$$

From (3) and (4):

$$\frac{w_i}{p_x} = \frac{U_y^i}{U_x^i} \tag{6}$$

From (2) and (4):

$$U_y^i = \lambda_i (p_y - \sum_{j \neq i} w_j) \tag{7}$$

$$\lambda_i = \frac{U_x^i}{p_x} \tag{8}$$

Therefore:

$$U_y^i = \frac{U_x^i}{p_x} (p_y - \sum_{j \neq i} w_j) \tag{9}$$

From (6), changing i to j:

$$w_j = p_x \frac{U_y^j}{U_x^j} \tag{10}$$

Substituting (10) into (9):

$$U_y^i = \frac{U_x^i}{p_x} \left[p_y - \sum_{j \neq i} \left(p_x \frac{U_y^j}{U_x^j} \right) \right] \tag{11}$$

Rearranging (11):

$$\frac{U_y^i}{U_x^i} = \frac{p_y}{p_x} - \sum_{j \neq i} \left(\frac{U_y^j}{U_x^j} \right) \tag{12}$$

Therefore:

$$\frac{p_y}{p_x} = \sum_{j=1}^{n} \left(\frac{U_y^j}{U_x^j} \right) \tag{13}$$

(13) defines the Lindahl-Samuelson efficiency conditions for a public goods equilibrium. Thus, from (6) and (13), we know that $w_i = v_i$, the "true" marginal

valuation for i, where $v_i(Y,x_i,p_x) = \dfrac{U_y^i\ (Y,x_i)}{U_x^i\ (Y,x_i)}\ p_x$ (14)

The mechanism only displays this optimality property at an equilibrium when participants behave competitively, however; and, since each individual faces a different ticket price, participants are never forced to behave competitively. This is the monopsony problem discussed by Arrow (1969) and formalized by Hurwicz (1972) in his work on the incentive compatibility of the tâtonnement mechanism for allocating private goods. As Hurwicz shows for that mechanism, unless the equilibrium price ratio is chosen on the first interation by a Walrasian auctioneer, players of a competitive game have an incentive to lie about their true demands. With private goods, this incentive goes to zero as n gets large (Roberts and Postlewaite, 1976), but that does not happen in our linear pricing model.

To illustrate the persistence of monopsony, consider an n-person model in which each participant determines his personalized ticket price by announcing a marginal valuation, w_i. Assuming downward sloping demands for the public good, each participant realizes that by increasing his marginal valuation (and thus his subsidy to other participants' subscriptions) he increases not only the equilibrium number of subscriptions to the public good, but also his ticket price for both the marginal and all inframarginal units. To determine his personalized ticket price, w_i, participant i assumes a demand function for the public good and a ticket price by each other person. Thus, t_i, (the number of tickets bought by i, representing, in equilibrium, the number of units of the public good purchased by i≠j) is now a function of w, the personalized ticket price:

$$t_i = t_i\ (w)$$

Participant i chooses a marginal valuation, w_i, to maximize utility, and demands $t_i\ (w_i)$ tickets.

Replacing t_i with t_i (w) in equation (1):

$$L_i = U^i(t_i(w) + y_i, x_i) + \lambda_i(M_i - p_y y_i - p_x x_i - w_i t_i(w) + y_i \sum_{j \neq i} w_j) \qquad (15)$$

The first order conditions are:

$$\frac{\partial L_i}{\partial y_i} = U^i_y - \lambda_i(p_y - \sum_{j \neq i} w_j) = 0 \qquad (16)$$

$$\frac{\partial L_i}{\partial w_i} = U^i_y \frac{\partial t_i}{\partial w} - \lambda_i \left[t_i(w) + w_i \frac{\partial t_i}{\partial w} \right] = 0 \qquad (17)$$

$$\frac{\partial L_i}{\partial x_i} = U^i_x - \lambda_i p_x = 0 \qquad (18)$$

$$\frac{\partial L_i}{\partial \lambda_i} = M_i - p_y y_i - p_x x_i - w_i t_i(w) + y_i \sum_{j \neq i} w_j = 0 \qquad (19)$$

From (17) and (18) we can solve for w_i, the reported marginal valuation for tickets:

$$\frac{U^i_y}{U^i_x} = \frac{t_i(w)}{p_x \frac{\partial t_i}{\partial w}} + \frac{w_i}{p_x} \qquad (20)$$

therefore:

$$w_i = p_x \frac{U^i_y}{U^i_x} - \frac{t_i}{\frac{\partial t_i}{\partial w}} \qquad (21)$$

Multiplying $\frac{t_i}{\frac{\partial t_i}{\partial w}}$ by $\frac{w_i}{w_i}$:

$$w_i = p_x \frac{U^i_y}{U^i_x} - \frac{t_i}{\frac{\partial t_i}{\partial w}} \cdot \frac{w_i}{w_i} \qquad (22)$$

therefore:

$$w_i = p_x \frac{U^i_y}{U^i_x} - w_i\left(\frac{1}{\varepsilon^i_t}\right), \text{ where} \tag{23}$$

$$\varepsilon^i_t = \frac{\partial t_i}{\partial w} \cdot \frac{w_i}{t_i} = \text{ the elasticity of equilibrium}$$

demands for the public good by all $j \neq i$ with respect to a change in w_i, i's reported marginal valuation.

Rearranging (23):

$$w_i = p_x \frac{U^i_y}{U^i_x}\left(\frac{\varepsilon^i_t}{1+\varepsilon^i_t}\right) = v_i\left(\frac{\varepsilon^i_t}{1+\varepsilon^i_t}\right) \tag{24}$$

Thus, i only reports his true marginal valuation, $v^i = \frac{U^i_y}{U^i_x} p_x$, when the aggregate elasticity of demand is infinite. Otherwise, $w_i < v_i$, implying that i underreports relative to his "true" marginal valuation.

B. <u>Non-linear Pricing Scheme</u>:

We turn now to a model which allows non-linear, government-subsidized pricing functions. In this model each buyer pays the government for personalized public goods tickets along a personalized non-linear pricing function, $w_i(t)$. Each seller of tickets, however, receives from the government each buyer's marginal valuation for each personalized ticket the seller sells to that buyer. To balance the budget we charge each participant a lump sum tax of $\frac{1}{n-1}$ times the budget deficit generated by all the other participants.

The "market" is organized as follows. The government assigns a personalized non-linear pricing function, $w_i(t)$ to each participant and "initializes" the market with a set of t_j's and $w_j(t_j)$'s for each $j \neq i$ and a set of lump sum taxes to use as parameters for the first round. To purchase t_i personalized tickets, i pays $\int_0^{t_i} w_i(t)dt$. Individuals still subscribe to units of the public good at marginal cost, p_y. Each individual maximizes utility by choosing personalized public goods tickets and private goods, and by subscribing to units of the public good, given p_y, p_x, m_i, $w_i(t)$, i's lump sum tax, and the t_j's and $w_j(t_j)$'s reported by the other participants. The government then agrees to pay each ticket seller each buyer's stated marginal valuation, $w_j(t_j)$, for each personalized ticket that buyer has demanded. Each buyer pays the government according to his personalized non-linear pricing schedule.

Participant i's problem is:

$$\text{Max } U^i(t_i + y_i, x_i)$$

$$\text{subject to: } M_i = p_x x_i + \left[p_y - \sum_{j\neq i} w_j(t_j)\right] y_i + \int_0^{t_i} w_i(t)dt + \frac{1}{n-1}\sum_{j\neq i}\left[w_j(t_j)\cdot t_j - \int_0^{t_j} w_j(t)dt\right]$$

The first order conditions are:

$$\frac{\partial L_i}{\partial y_i} = U_y^i - \lambda_i\left[p_y - \sum_{j\neq i} w_j(t_j)\right] = 0 \tag{25}$$

$$\frac{\partial L_i}{\partial t_i} = U_y^i - \lambda_i w_i(t_i) = 0 \tag{26}$$

$$\frac{\partial L_i}{\partial x_i} = U_x^i - \lambda_i p_x = 0 \tag{27}$$

$$\frac{\partial L_i}{\partial \lambda_i} = M_i - p_x x_i - \left[p_y - \sum_{j\neq i} w_j(t_j)\right] y_i - \int_0^{t_i} w_i(t)dt - \frac{1}{n-1}\sum_{j\neq i}\left[w_j(t_j)\cdot t_j - \int_0^{t_j} w_j(t)dt\right] = 0 \tag{28}$$

From (25) and (26) we can solve for $w_i(t_i)$, the marginal valuation along $w_i(t)$ implied by a choice of t_i:

$$w_i(t_i) = \frac{U_y^i}{U_x^i} p_x \tag{29}$$

Therefore, from (10), (14) and (29), we know that,

$$w_i(t_i) = v_i(Y, x_i, p_x) \tag{30}$$

the "true" marginal valuation.

The non-linear pricing scheme for personalized tickets eliminates the monopsonistic incentive for i to underreport relative to his true marginal valuation, by making him able to behave like a discriminating monopsonist. The monopsonist treats the non-linear price function as the supply function for personalized tickets and picks the marginal ticket price which corresponds to his true marginal valuation since he does not have to pay for all personalized tickets at that price. Like the discriminating monopsonist, he can pay for each unit at the limit price for that unit along the supply function for tickets. He only pays his marginal valuation for the last unit he purchases.

While this scheme solves the classic monopsony problem which comes when all units must be purchased at the same price, it does not preclude the kind of manipulation discussed by Hurwicz (1972) and Green and Laffont (1979). As long as participants exhibit myopic Cournot-Nash behavior (i.e. take as given the behavior of other participants), the Nash equilibrium satisfies the Lindahl-Samuelson conditions. If, however, participants can affect the equilibrium outcome to their own benefit by playing non-Nash strategies, there continues to be an incentive to misrepresent preferences. This incentive goes to zero an n gets large under the competitive mechanism for allocating private goods because all participants face the same price ratios (Roberts and Postlewaite, 1976). If, somehow, all participants in a public goods allocation process were to face the same non-linear price function, we conjecture that that incentive to misrepresent would also go to zero as n became large.

For example, suppose the non-linear pricing function, which applied to all participants, was a linear function of t with slope α. The auctioneer would only have to change α if any adjustment were needed. Thus,

$$w_i(t) = \alpha t_i \quad \forall i, \tag{31}$$

$$\int_0^{t_i} w_i(t)dt = \frac{\alpha t_i^2}{2} \quad \forall i, \text{ and}$$

$$w_i(t_i) = \alpha t_i \tag{32}$$

substituting (31) and (32) into the budget constraint, participant i's problem becomes:

$$\text{Max } U^i(y_i + t_i, x_i)$$

$$\text{subject to: } M_i = p_x x_i + (p_y - \sum_{j\neq i} \alpha t_j)y_i + \frac{\alpha t_i^2}{2} + \frac{1}{n-1} \sum_{j\neq i} \frac{\alpha t_j^2}{2}$$

The first order conditions are:

$$\frac{\partial L_i}{\partial y_i} = U_y^i - \lambda_i(p_y - \sum_{j\neq i} \alpha t_j) = 0 \tag{33}$$

$$\frac{\partial L_i}{\partial t_i} = U_y^i - \lambda_i \alpha t_i = 0 \tag{34}$$

$$\frac{\partial L_i}{\partial x_i} = U_x^i - \lambda_i p_x = 0 \tag{35}$$

$$\frac{\partial L_i}{\partial \lambda_i} = M_i - p_x x_i - y_i \left(p_y - \sum_{j\neq i} \alpha t_j\right) - \frac{\alpha t_i^2}{2} -$$

$$\frac{1}{n-1} \sum_{j\neq i} \frac{\alpha t_j^2}{2} = 0 \tag{36}$$

From (33) and (34):

$$U_y^i - \frac{U_y^i}{\alpha t_i}\left(p_y - \sum_{j\neq i} \alpha t_j\right) = 0 \tag{37}$$

From (37) we can solve for :

$$p_y = \alpha t_i + \alpha \sum_{j\neq i} t_j = \alpha \sum_{i=1}^{n} t_i \tag{38}$$

Since $t_i = \sum_{j\neq i} y_j$ in equilibrium:

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \sum_{j\neq i} y_j = (n-1) \sum_{i=1}^{n} y_i \tag{39}$$

Substituting (39) into (38):

$$p_y = (n-1)\alpha \sum_{i=1}^{n} y_i \tag{40}$$

Therefore, since $Y = \sum_{i=1}^{n} y_i$ in equilibrium,

$$\alpha = \frac{p_y}{(n-1)Y} \tag{41}$$

Thus, if an equilibrium Y can be achieved through this mechanism, there is a single α which sustains it.

To show that the budget balances, we sum the incomes:

$$\sum_{i=1}^{n} M_i = p_X \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i(\alpha \sum_{i=1}^{n} t_i - \alpha \sum_{j \neq i} t_j) + \frac{\alpha}{2} \sum_{i=1}^{n} t_i^2$$

$$+ \frac{n-1}{n-1} \frac{\alpha}{2} \sum_{i=1}^{n} t_i^2$$

$$= p_X \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} t_i y_i + \alpha \sum_{i=1}^{n} t_i^2$$

$$= p_X \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} y_i \sum_{j \neq i} y_j + \alpha \sum_{i=1}^{n} (\sum_{j \neq i} y_j)^2$$

$$= p_X \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} \sum_{j \neq i} y_j (y_i + \sum_{j \neq i} y_j)$$

$$= p_X \sum_{i=1}^{n} x_i + \alpha \sum_{i=1}^{n} \sum_{j \neq i} y_j (y_i + t_i)$$

$$= p_X \sum_{i=1}^{n} x_i + \alpha \ (n-1) \sum_{i=1}^{n} y_i \quad Y$$

Therefore, from equation (40):

$$\sum_{i=1}^{n} M_i = p_X \sum_{i=1}^{n} x_i + p_y Y \tag{42}$$

Therefore, the total of consumer incomes equals total expenditures on private goods and the public good. There is no deficit or surplus.

For 3 individuals, the equilibrium and budget balance conditions are illustrated in Figure 1. $Y = 15$ and $\alpha = p_y/30$ represent a public goods equilibrium and non-linear price function which satisfy the Lindahl-Samuelson conditions.

In figure 1 each person pays his or her shaded area for tickets and subscription, leaving the white area as deficit. The subscription and ticket payments plus the deficits equal the total cost of building the public good. The lump sum taxes are:

1 pays 1/2 (2's deficit + 3's deficit)

2 pays 1/2 (1's deficit + 3's deficit)

3 pays 1/2 (1's deficit + 2's deficit)

We now show that the maximization problem for participant i satisfies the 2nd order conditions for constrained utility maximization. Totally differentiating equations (33)-(36) and writing them in matrix form, the left-hand side is:

$$\begin{bmatrix} U^i_{xx} & U^i_{xy} & U^i_{xy} & -p_x \\ U^i_{yx} & U^i_{yy} & U^i_{yy} & -(p_y - \alpha \sum_{j=i} t_j) \\ U^i_{yx} & U^i_{yy} & (U^i_{yy} - \lambda_i \alpha) & -\alpha t_i \\ -p_x & -(p_y - \alpha \sum_{j \neq i} t_j) & -\alpha t_i & 0 \end{bmatrix} \begin{bmatrix} dx_i \\ dy_i \\ dt_i \\ d\lambda_i \end{bmatrix}$$

Now, substitute αt_i for $(p_y - \alpha \sum_{j \neq i} t_j)$ and consider the bordered principal minor:

$$\begin{bmatrix} U^i_{yy} & U^i_{yy} & -\alpha t_i \\ U^i_{yy} & (U^i_{yy} - \lambda_i \alpha) & -\alpha t_i \\ -\alpha t_i & -\alpha t_i & 0 \end{bmatrix}$$

It's determinant is:

$$-\alpha t_i \left[\alpha t_i (U^i_{yy} - \lambda_i \alpha - \alpha t_i\ U^i_{yy}\right] +$$

$$\alpha t_i(\alpha t_i\ U^i_{yy} - \alpha t_i\ U^i_{yy}) = \alpha^3 t_i{}^2 \lambda_i$$

Since λ_i, the marginal utility of M_i, is greater than zero,

$$\alpha^3 t_i{}^2 \lambda_i > 0$$

Now consider Δ, the determinant of the coefficient matrix itself. Expanding along the 4th row and collecting terms:

$$\Delta = \lambda_i \alpha \left[U^i_{yy}\ p_x{}^2 - 2U^i_{xy}\ p_x \alpha t_i + U^i_{xx}\ (\alpha t_i)^2\right]$$

Since p_x and αt_i are prices, the quadratic form in brackets is negative by the assumption of quasi-concavity of the utility function. Therefore,

$$\Delta < 0 \text{ if } U(x_i, y_i + t_i) \text{ is quasi-concave}$$

Thus, the 4X4 bordered Hessian has sign $(-1)^3$ and the relevant principal minor alternates in sign.

Figure 2 illustrates why the second order conditions hold. The curve with intercepts $\frac{M_i}{p_x}$ and $\left(\frac{2M_i}{\alpha}\right)^{1/2}$ represents the participant's budget line if he buys only tickets:

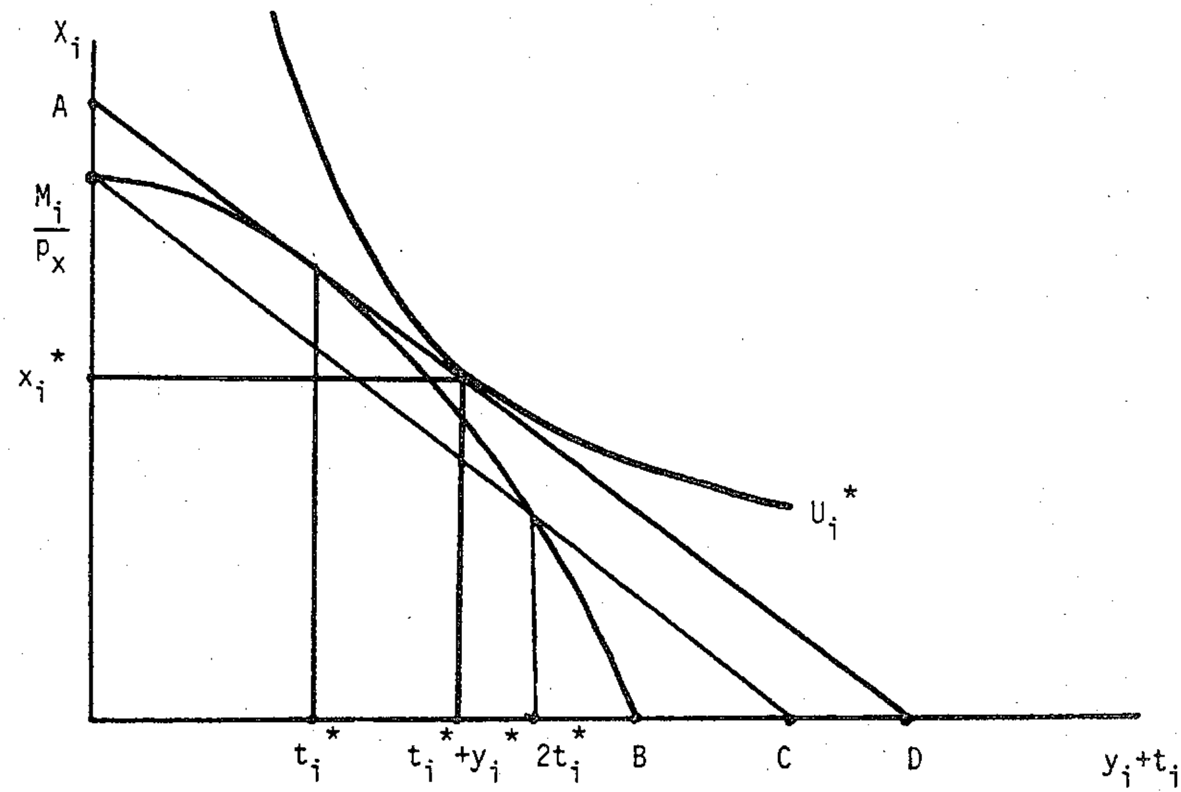
$$M_i = p_x x_i + \frac{\alpha t_i{}^2}{2}$$

The linear function with intercepts $\frac{M_i}{p_x}$ and $\frac{M_i}{(p_y - \sum_{j \neq i} \alpha t_j)}$ represents the participant's budget line if buys only subscription units:

$$M_i = p_x x_i + (p_y - \sum_{j \neq i} \alpha t_j) y_i$$

Figure 2

Utility Maximization with Unique "Linear" Non-linear Ticket Price Function



$$A = \frac{M_i + \frac{\alpha t_i^{*2}}{2}}{p_x}$$

$$B = \left(\frac{2M_i}{\alpha}\right)^{1/2}$$

$$C = \frac{M_i}{p_y - \sum_{j \neq i} \alpha t_j}$$

$$D = \frac{M_i^* + \frac{\alpha t_i^{*2}}{2}}{p_y - \sum_{j \neq i} \alpha t_j}$$

The participant actually purchases along the ticket budget line as long as the marginal cost of a ticket is less than the marginal cost of a subsidized unit. By buying along the ticket budget line, however, the participant gets a subsidy from reselling the tickets equal to $\frac{\alpha t_i^{*2}}{2}$. In fact, allowing participants to demand negative y_i's, so $y_i^* + t_i^*$ is to the left of t_i^*, they can buy t_i^* and sell y_i^* back to the government at the subsidized price. In that case the budget line will be the linear function with intercepts

$$\frac{M_i + \frac{\alpha t_i^{*2}}{2}}{p_x} \quad \text{and} \quad \frac{M_i + \frac{\alpha t_i^{*2}}{2}}{(p_y - \sum_{j\neq i} \alpha t_j)} \; :$$

$$M_i = p_x x_i + (p_y - \sum_{j\neq i} \alpha t_j)\, y_i - \frac{\alpha t_i^{*2}}{2}$$

This function is just tangent to the ticket budget line at t_i^* . The lump sum tax is not included because it simply represents a parallel shift in all three budget equations. Thus, as long as negative y_i^*'s are allowed, the participant can choose any point along the augmented budget line. Quasi-concavity of the utility function ensures that such a maximizing choice exists.

The question of whether such a public goods equilibrium can be sustained by one equilibrium non-linear pricing function for all participants remains an open question. As we show below, we know it exists for some classes of utility functions and income distributions, but we cannot yet present a general existence proof.

IV. Examples

A. n-people, Identical Cobb-Douglas Utility Functions, Unequal Incomes

Let $U_i = \beta \ln (y_i + t_i) + (1 - \beta) \ln x_i$

The Lagrange equation is:

$$L_i = \beta \ln (y_i + t_i) + (1 - \beta) \ln x_i + \lambda_i (M_i - (p_y - \alpha \sum_{j \neq i} t_j) y_i - \frac{\alpha t_i^2}{2}$$

$$- p_x x_i - \left(\frac{1}{n-1}\right) \frac{\alpha}{2} \sum_{j \neq i} t_j^2) \tag{43}$$

The relevant first order conditions are:

$$\frac{\partial L_i}{\partial t_i} = \frac{\beta}{y_i + t_i} - \lambda_i \alpha t_i = 0 \tag{44}$$

$$\frac{\partial L_i}{\partial x_i} = \frac{(1 - \beta)}{x_i} - \lambda_i p_x = 0 \tag{45}$$

From (44) and (45):

$$p_x x_i = \frac{(1 - \beta)}{\beta} \alpha t_i (y_i + t_i) \tag{46}$$

substituting Y^* for $y_i + t_i$ in (46):

$$p_x x_i = \frac{(1 - \beta)}{\beta} \alpha t_i Y^* \tag{47}$$

Summing over i:

$$p_x \sum_{i=1}^{n} x_i = \frac{(1 - \beta)}{\beta} \alpha Y^* \sum_{i=1}^{n} t_i \tag{48}$$

Substituting $(n-1) Y^*$ for $\sum_{i=1}^{n} t_i$ in (48):

$$p_x \sum_{i=1}^{n} x_i = \frac{\alpha(1 - \beta)}{\beta} (Y^*)^2 (n-1) \tag{49}$$

Substituting $\frac{p_y}{(n-1)\alpha}$ for Y^* in (49):

$$p_x \sum_{i=1}^{n} x_i = \frac{(1 - \beta)}{\beta} \frac{p_y^2}{\alpha(n-1)} \tag{50}$$

Substituting (50) and $\frac{p_y}{(n-1)\alpha}$ for Y* into (42):

$$\sum_{i=1}^{n} M_i = \frac{(1-\beta)}{\beta} \frac{p_y^2}{(n-1)\alpha} + \frac{p_y^2}{\alpha(n-1)}$$

Therefore:

$$\sum_{i=1}^{n} M_i = \frac{1}{\beta} \frac{p_y^2}{\alpha(n-1)}$$

Therefore:

$$\alpha = \frac{p_y^2}{(n-1)\beta \sum_{i=1}^{n} M_i} \quad \text{in equilibrium} \tag{51}$$

The solution is so simple in this case because, with identical Cobb-Douglas utility functions, each participant's income effects are exactly offset by the income effects of all the other participants. Consequently, the equilibrium α only reflects the sum of the incomes and not the income distribution. As the next example shows, the extension to different Cobb-Douglas utility functions is quite complicated, even if we restrict ourselves to only 2 participants.

B. <u>2-People, Different Cobb-Douglas Utility Functions, Different Incomes</u>

Let $U_1 = \beta_1 \ln(y_1 + t_1) + (1 - \beta_1)\ln x_1$

$U_2 = \beta_2 \ln(y_2 + t_2) + (1 - \beta_2)\ln x_2$

From (47) we know:

$$p_x x_1 = \frac{(1-\beta_1)}{\beta_1} \alpha t_1 Y^* \tag{52}$$

$$p_x x_2 = \frac{(1-\beta_2)}{\beta_2} \alpha t_2 Y^* \tag{53}$$

The budget constraints are:

$$M_1 = p_x x_1 + \alpha t_1 y_1 + \frac{\alpha t_1^2}{2} + \frac{\alpha t_2^2}{2}$$

$$M_2 = p_x x_2 + \alpha t_2 y_2 + \frac{\alpha t_1^2}{2} + \frac{\alpha t_2^2}{2}$$

Therefore:

$$M_1 = p_x x_1 + \frac{\alpha}{2} (t_1 + t_2)^2$$

$$M_2 = p_x x_2 + \frac{\alpha}{2} (t_1 + t_2)^2$$

Substituting Y* for <u>one</u> $(t_1 + t_2)$ in each equation

$$M_1 = p_x x_1 + \frac{\alpha}{2} Y^* (t_1 + t_2) \tag{54}$$

$$M_2 = p_x x_2 + \frac{\alpha}{2} Y^* (t_1 + t_2) \tag{55}$$

Substituting (52) and (53) in (54) and (55), respectively:

$$M_1 = \alpha Y^* \left[\frac{(1 - \beta_1)}{\beta_1} t_1 + \frac{(t_1 + t_2)}{2} \right]$$

$$M_2 = \alpha Y^* \left[\frac{(1 - \beta_2)}{\beta_2} t_2 + \frac{(t_1 + t_2)}{2} \right]$$

Substituting $\frac{p_y}{(n-1)\alpha}$ for Y*:

$$(n-1) \frac{M_1}{p_y} = \frac{(1 - \beta_1)}{\beta_1} t_1 + \frac{t_1 + t_2}{2} \tag{56}$$

$$(n-1) \frac{M_2}{p_y} = \frac{(1 - \beta_2)}{\beta_2} t_2 + \frac{t_1 + t_2}{2} \tag{57}$$

Now, let:

$$B_1 = \frac{(1 - \beta_1)}{\beta_1} \tag{58}$$

$$B_2 = \frac{(1 - \beta_2)}{\beta_2} \tag{59}$$

Noting (n-1=1), substituting (58) into (56) and (59) into (57) and collecting terms:

$$t_1 = \frac{2M_1 - p_y t_2}{p_y(2B_1 + 1)} \tag{60}$$

$$t_2 = \frac{2M_2 - p_y t_1}{p_y(2B_2 + 1)} \tag{61}$$

Substituting (60) into (61) and collecting terms:

$$t_2 = \frac{2M_2 [(2B_1 + 1) - M_1]}{p_y[(2B_1 + 1)(2B_2 + 1)-1]} \tag{62}$$

Substituting (62) into (60) and collecting terms:

$$t_1 = \frac{2M_1 [(2B_2 + 1) - M_2]}{p_y[(2B_1 + 1)(2B_2 + 1)-1]} \tag{63}$$

Substituting (63) and (64) into $\alpha = \dfrac{p_y}{(t_1 + t_2)}$ and collecting terms:

$$\alpha = \frac{p_y^2(2B_1B_2 + B_1 + B_2)}{2(M_1B_2 + M_2B_1)} \tag{64}$$

Now if we compare (64) and (51) we can see the different income effects which come from having different Cobb-Douglas utility functions. Whereas in (51) we just sum the incomes in arriving at an equilibrium α, in (64) we are forced to derive a complex weighted sum of the incomes, where the weights are the Cobb-Douglas exponents.

V. Implementing the Non-linear Unique α Ticket Mechanism

A. The Dynamic Adjustment Process

While we have not been able to solve for explicit equilibrium α functions for the general n-person case with income effects and different preferences, we can show by computer simulation that a simple Walrasian adjustment mechanism finds a fixed point with 3 individuals who have different Cobb-Douglas utility functions and different incomes. Generalizing equation (43), the Lagrange problem for participant i is:

$$L_i = (1 - \beta_i) \ln x_i + \beta_i \ln (y_i + t_i) + \lambda_i \Big[M_i - p_x x_i - (p_y - \alpha \sum_{j \neq i} t_j) y_i - \frac{\alpha t_i^2}{2} - \frac{1}{n-1} \sum_{j \neq i} \frac{\alpha t_j^2}{2} \Big] \tag{65}$$

The first order conditions are:

$$\frac{\partial L_i}{\partial x_i} = \frac{1 - \beta_i}{x_i} - \lambda_i p_x = 0 \tag{66}$$

$$\frac{\partial L_i}{\partial t_i} = \frac{\beta_i}{y_i + t_i} - \lambda_i \alpha t_i = 0 \tag{67}$$

$$\frac{\partial L_i}{\partial y_i} = \frac{\beta_i}{y_i + t_i} - \lambda_i (p_y - \alpha \sum_{j \neq i} t_j) = 0 \tag{68}$$

$$\frac{\partial L_i}{\partial \lambda_i} = M_i - p_x x_i - y_i (p_y - \alpha \sum_{j \neq i} t_j) - \frac{\alpha t_i^2}{2} - \frac{1}{n-1} \sum_{j \neq i} \frac{\alpha t_j^2}{2} = 0 \tag{69}$$

Equations (66) - (69) allow us to solve for i's demand function, $y_i(M_i, \alpha, p_x, p_y, t_j)$:

$$y_i = \frac{\beta_i \left[M_i - \frac{\alpha}{2}\left(\frac{p_y}{\alpha} - \sum_{j\neq i} t_j\right)^2 - \frac{\alpha}{2(n-1)} \sum_{j\neq i} t_j^2\right]}{p_y - \alpha \sum_{j\neq i} t_j}$$

$$- (1 - \beta_i)\left(\frac{p_y}{\alpha} - \sum_{j\neq i} t_j\right) \tag{70}$$

The price of x does not enter the actual function because cross elasticities are zero for Cobb-Douglas utility functions.

Now, consider the following adjustment process. We initialize by letting t_i's be equal across all i and y_i's be equal across all i (i.e. $t_i^o = \ldots = t_n^o$ and $y_i^o = \ldots = y_n^o$). We know from equation (41) that

$$\alpha = \frac{p_y}{\frac{(n-1)}{n} \sum_{i=1}^{n} y_i} = \frac{p_y}{(n-1)(y_i + t_i)} = \frac{p_y}{(n-1)Y}$$

Therefore, let Y be the average Y indicated above to initialize α:

$$\alpha^o = \frac{p_y}{\frac{(n-1)}{n} \sum_{i=1}^{n} (y_i + t_i)}$$

In period 1, participant i uses these initial values to solve his maximization problem, choosing y_i^1 and t_i^1:

$$y_i^1 = \frac{\beta_i \left[M_i - \frac{\alpha^o}{2}\left(\frac{p_y}{\alpha^o} - \sum_{j\neq i} t_j^o\right)^2 - \frac{\alpha^o}{2(n-1)} \sum_{j\neq i} (t_j^o)^2\right]}{\left(p_y - \alpha^o \sum_{j\neq i} t_j^o\right)}$$

$$- (1 - \beta_i)\left(\frac{p_y}{\alpha^o} - \sum_{j\neq i} t_j^o\right) \tag{71}$$

$$t_i^{\,1} = \frac{p_y}{\alpha^0} - \sum_{j\neq i} t_j^{\,0} \tag{72}$$

Thus, $t_i^{\,1}$ represents i's demand for subscription tickets and $y_i^{\,1}$ represents i's supply of subscription tickets. If $t_i^{\,1} \neq \sum_{j\neq i} y_j^{\,1}$, then i's ticket market is not in equilibrium and the subsidy reported to each $j\neq i$ should be changed. If $t_i^{\,1} > \sum_{j\neq i} y_j^{\,1}$, it implies demand exceeds supply and the subsidy to each $j\neq i$ should be raised. If $t_i^{\,1} < \sum_{j\neq i} y_j^{\,1}$ the subsidy should be reduced.

Let $\tilde{t}_i^{\,1} = t_i^{\,1} - \gamma(\sum_{j\neq i} y_j^{\,1} - t_i^{\,1}) \quad \forall i \qquad 0 < \gamma < 1$

Now suppose each i's ticket market is out of equilibrium and $\tilde{t}_i^{\,1}$ is computed for each i. We can now report an adjusted subsidy to each i as a function of each $\tilde{t}_j^{\,1}$, $j\neq i$:

$\tilde{S}_i^{\,1} = \alpha^0 \sum_{j\neq i} \tilde{t}_j^{\,1}$ = subsidy reported to i on the 2nd iteration

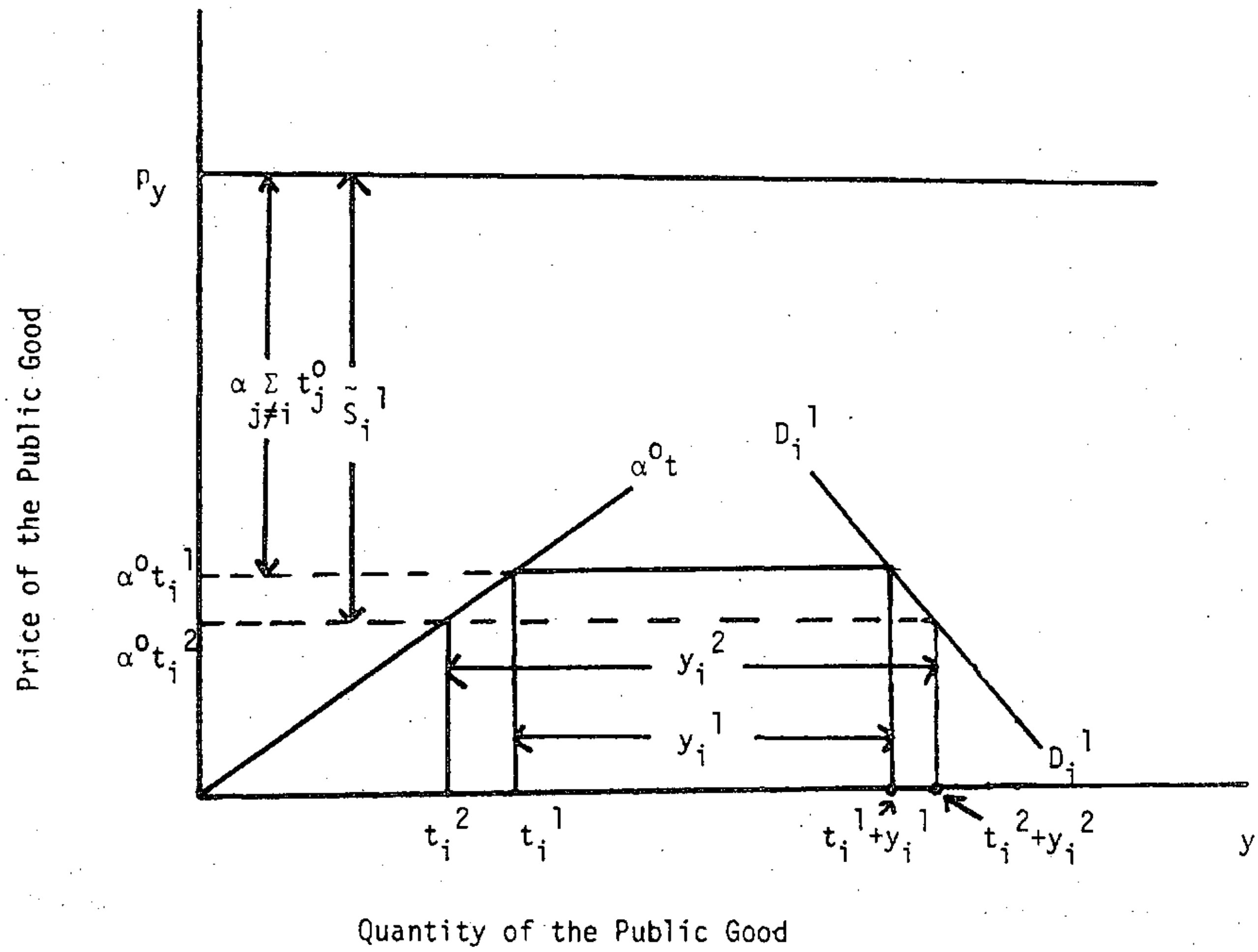
Figure 3 illustrates the effect on i's demand for tickets and supply of subsidized units if $\tilde{S}_i > \alpha^0 \sum_{j\neq i} t_j^{\,1}$. Assuming the income effect of increasing i's lump sum tax when $\tilde{S}_i$ is raised is small, he supplies more subscription units ($y_i^{\,2} > y_i^{\,1}$), which is what we wanted, since the demand for tickets by $j\neq i$ exceeded i's supply. At the same time, however, the increase in i's subsidy reduces the quantity of tickets demanded ($t_i^{\,2} < t_i^{\,1}$) and increases the total quantity of the public good i demands ($t_i^{\,2} + y_i^{\,2} > t_i^{\,1} + y_i^{\,1}$).

The utility maximizing values $y_i^{\,1}$ and $t_i^{\,1}$ also imply a new estimate for α:

$$\alpha^1 = \frac{p_y}{\frac{(n-1)}{n} \sum_{i=1}^{n} (t_i^{\,1} + y_i^{\,1})}$$

Figure 3

Adjustment of Subsidized Subscription Prices



DiDi = i's demand curve for the public good

However, there is a tendency for these α estimates to promote divergence rather than convergence to equilibrium. The problem is that if α^0 is too small, participants begin by demanding too much of the public good. That will cause α^1 to be even smaller than α^0, encouraging a further increase in the quantity demanded of the public good. If α^0 is too large, the reverse problem occurs: initial demand for the public good is too small and α^1 will be even larger. We counteract this tendency by adjusting α^0 in the opposite direction from the movement implied by our estimate of α^1. Now let,

$$\tilde{\alpha}^1 = \alpha^0 - \delta(\alpha^1 - \alpha^0), \qquad 0 < \delta < 1$$

Thus, if $\alpha^1 > \alpha^0$, $\tilde{\alpha}^1 < \alpha^1$ and if $\alpha^1 < \alpha^0$, $\tilde{\alpha}^1 > \alpha^1$.

To begin round 2, participant i is given the following parameters: $\tilde{\alpha}^1$, $\tilde{S}_i{}^1$, and $\tilde{T}_i{}^1$, where,

$$\tilde{S}_i{}^1 = \tilde{\alpha}^1 \sum_{j\neq i} \tilde{t}_j{}^1 \tag{73}$$

$$\tilde{T}_i{}^1 = \frac{\tilde{\alpha}^1}{2(n-1)} \sum_{j\neq i} (\tilde{t}_j{}^1)^2 = \text{i's new lump sum tax.} \tag{74}$$

Substituting (73) and (74) into (71) and (72), i's demand functions for round 2 are:

$$y_i{}^2 = \frac{\beta_i \quad M_i - \frac{1}{2}(p_y - \tilde{S}_i{}^1)^2 - \tilde{T}_i{}^1}{(p_y - \tilde{S}_i{}^1)}$$

$$- \frac{(1 - \beta_i)}{\tilde{\alpha}^1}(p_y - \tilde{S}_i{}^1)$$

$$t_i{}^2 = \frac{1}{\tilde{\alpha}^1}(p_y - \tilde{S}_i{}^1)$$

By induction, therefore, at iteration m:

$$\tilde{t}_i^{\,m} = t_i^{\,m} - \gamma\Big(\sum_{j\neq i} y_j^{\,m} - t_i^{\,m}\Big),$$

$$\tilde{\alpha}^m = \tilde{\alpha}^{m-1} - \delta(\alpha^m - \tilde{\alpha}^{m-1}),$$

$$\tilde{S}_i^{\,m} = \tilde{\alpha}^m \sum_{j\neq i} \tilde{t}_j^{\,m},$$

$$\tilde{T}_i^{\,m} = \frac{\tilde{\alpha}^m}{2(n-1)} \sum_{j\neq i} (\tilde{t}_j^{\,m})^2,$$

$$y_i^{\,m} = \frac{\beta_i \quad M_i - \frac{1}{2} \;(p_y - \tilde{S}_i^{\,m-1})^2 - \tilde{\tilde{T}}_i^{\,m-1}}{(p_y - \tilde{S}_i^{\,m-1})} - \frac{(1 - \beta_i)}{\tilde{\alpha}^{m-1}}\left(p_y - \tilde{S}_i^{\,m-1}\right), \text{ and}$$

$$t_i^{\,m} = \frac{1}{\tilde{\alpha}^{m-1}}(p_y - \tilde{S}_i^{\,m-1}).$$

The process continues until a fixed point is reached.

B. <u>Computer Simulation</u>

We ran a computer simulation of the above adjustment process for 3 participants with Cobb-Douglas utility functions. The utility and income parameters were:

$\beta_1 = .12$

$\beta_2 = .10$

$\beta_3 = .08$

$M_1 = 80$

$M_2 = 120$

$M_3 = 100$

After some trial and error we discovered that the process converged better if γ and δ were small to begin with (otherwise the process sometimes diverges at first). Once convergence was underway we could increase γ and δ and increase the speed of convergence without disturbing the equilibration process. So, we assigned γ and δ as follows:

$$\gamma = \begin{cases} .05 & \text{if } m < 25 \\ .20 & \text{if } m \geq 25 \end{cases}$$

$$\delta = \begin{cases} .05 & \text{if } m < 25 \\ .20 & \text{if } m \geq 25 \end{cases}$$

We ran the process with the above parameters from four different consistent starting α^o, t_i^o, y_i^o combinations:

1. $\alpha^o = \frac{1}{30}$, $t_i^o = 10$, $y_i^o = 5$
2. $\alpha^o = \frac{1}{60}$, $t_i^o = 20$, $y_i^o = 10$
3. $\alpha^o = \frac{1}{90}$, $t_i^o = 30$, $y_i^o = 15$
4. $\alpha^o = \frac{1}{120}$, $t_i^o = 40$, $y_i^o = 20$

All four runs converged to the same fixed point:

$$\alpha = .016875$$
$$y_1 = 10.55, \quad t_1 = 19.08 = 5.13 + 13.95$$
$$y_2 = 5.13, \quad t_2 = 24.50 = 10.55 + 13.95$$
$$y_3 = 13.95, \quad t_3 = 15.68 = 10.55 + 5.13$$
$$Y = 29.63 = 10.55 + 19.08 = 5.13 + 24.50 = 13.95 + 15.68$$

The number of iterations necessary in each case was:

1. 61
2. 60
3. 72
4. 80

We have also tried a variety of other parametric combinations that converged. Some examples are reported in Table 1. Notice that different income distributions do generate different Lindahl prices and different equilibrium Y's and α's. In each case $y_i + t_i = Y$ for each i, $Y = \sum_{i=1}^{n} y_i$ and the payments sum to $p_y\, y$, given $p_y = 1$. Thus, the computer simulations suggest that the unique α ticket mechanism can be implemented even when there are substantial income effects. Even though we do not yet understand the formal stability properties of this mechanism, it appears to work even in a dynamic setting.

However, somewhat surprisingly, the total payments are approximately equal across all participants in every example, even though the distribution of payments across tickets, subscription units and lump sum taxes changes with the income effects. This occurs because the person with the highest Lindahl price and, therefore, the highest non-linear payment, always pays the lowest subscription payment and lump sum tax; the person with the lowest Lindahl price pays the highest subscription and lump sum payments. These tend to offset one another, leaving the total payments approximately equal. Thus, this particular lump sum tax rule tends to be regressive. In fact, since in five out of the six cases the person with the highest marginal valuation pays the lowest total payment, the income redistribution implied by these lump sum taxes is almost perversely regressive.

If we find equal payments disturbing for their income redistribution effect, there are other incentive compatible lump sum taxes we can employ. For example,

$$\text{Let } T_i = \alpha \sum_{j=i} C_{ij} \frac{t_j^{\,2}}{2}, \text{ where}$$

$$C_{ij} = \frac{M_i^{\mu}}{\sum_{k \neq j} M_k^{\mu}}, \quad \mu \geq 0$$

TABLE 1
COMPUTER SIMULATION RESULTS

$U_i = \beta_i \ln(y_i + t_i) + (1-\beta_i) \ln x_i$ $\beta_1 = .12$ $\beta_2 = .10$ $\beta_3 = .08$ $P_x = 1;\ P_y = 1$

Example	i	Income	t_i	y_i	α	Lindahl Price	Non-linear Payment	Subscription Payment	Lump-sum Tax	Total Payment
1	1	80	19.163	10.037		.32813	3.14396	3.29359	3.29687	9.73442
	2	100	20.060	9.140	(.017123)	.34349	3.44518	3.13963	3.14626	9.73107
	3	120	19.177	10.023		.32837	3.14855	3.29140	3.29457	9.73452
				29.200						29.200
2	1	80	19.081	10.549		.32199	3.07196	3.39656	3.56926	10.03778
	2	120	24.498	5.132	(.016875)	.41340	5.06378	2.12152	2.57335	9.75865
	3	100	15.681	13.949		.26462	2.07473	3.69097	4.06787	9.83357
				29.630						29.630
3	1	120	30.107	.456		.49255	7.41461	.22459	1.96768	9.60688
	2	80	15.451	15.112	(.016360)	.25278	1.95284	3.81966	4.69857	10.47107
	3	100	15.568	14.995		.25469	1.98253	3.81879	4.68372	10.48504
				30.563						30.563
4	1	100	24.609	5.051		.41483	5.10432	2.09546	2.56518	9.76496
	2	80	15.591	14.069	(.016857)	.26282	2.04879	3.69794	4.09295	9.83968
	3	120	19.121	10.540		.32232	3.08157	3.39755	3.57656	10.05568
				29.660						29.660
5	1	100	24.445	6.065		.40060	4.89639	2.42959	3.04270	10.36868
	2	120	24.361	6.150	(.016388)	.39923	4.86280	2.45518	3.05949	10.37747
	3	80	12.215	18.295		.20018	1.22260	3.66206	4.87959	9.76425
				30.510						30.510
6	1	120	30.023	.959		.48451	7.27324	.46467	2.17546	9.91337
	2	100	19.785	11.197	(.016138)	.31929	3.15858	3.57537	4.23279	10.96674
	3	80	12.156	18.826		.19617	1.19234	3.69363	5.21591	10.10188
				30.982						30.982

Note that $\mu=0$ implies $C_{ij} = \frac{1}{n-1}$, the original weighting scheme. As μ increases higher income participants pay higher percentages of the lump sum taxes.

V. Conclusions and Suggestions for Further Research

This paper has shown that it is possible to construct a market-like mechanism for allocating public goods which is incentive compatible and relatively simple to implement. Because it only employs one common pricing parameter, it does not require the central authority to figure out personalized prices and it eliminates the monopsony element in personalized prices. It is not without its faults, however. Because of the lump sum taxes it might not always preserve individual rationality. In addition, because the person with the highest marginal valuation buys the most tickets and therefore generates the highest deficit to be shared by everyone else, the mechanism may perversely redistribute income. That is, if the rich have the highest marginal valuations, the poor will tend to pay the highest lump sum taxes.

At this point we also do not know whether a unique α exists in general or whether the adjustment mechanism is stable for any general set of parameters. We plan to continue theoretical work on existence and stability and to run human subject experiments to further test the applicability of the mechanism and adjustment process. In addition, we are working on a more generalized general equilibrium model and on less regressive means of balancing the budget.

REFERENCES

Arrow, K. J. (1969), "The Organization of Economic Activity: Issues Pertinent to the Choice of Market vs. Nonmarket Allocation," in U.S. Congress Joint Economic Committee, The Analysis and Evaluation of Public Expenditures: The PPB System, Vol. I, U.S. Government Printing Office, Washington, D.C.

Brubaker, E. R. (1975), "Free Ride, Free Revelation, or Golden Rule?," Journal of Law and Economics, 18(1): 147-161.

Dréze, J. and de la Valleé Poussin (1971), "A Tâtonnement Process for Public Goods," Review of Economic Studies, 38: 133-150.

Green, J. and Laffont, J. J. (1977), "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," Econometrica, 45: 472-538.

Groves, T. (1973), "Incentives and Teams," Econometrica, 41: 617-633.

Groves, T. and Ledyard, J. (1977), "Optimal Allocation of Public Goods: A Solution to the Free Rider Problem," Econometrica, 45: 783-810.

Hurwicz, L. (1979), "On Informationally Decentralized Systems," in R. Rodner and C. B. McGuire (eds.), Decision and Organization (Amsterdam: North Holland).

Kalai, E. (1981), "Contracts, Consensus, and Group Decisions," Northwestern University Center for Mathematical Studies in Economics and Management Sciences Discussion Paper #479.

Malinvand, E. (1971), "A Planning Approach to the Public Goods Problem," Swedish Journal of Economics, 1: 96-111.

MasColell, A. (1980), "Efficiency and Decentralization in the Pure Theory of Public Goods," Quarterly Journal of Economics, 94(4): 625-641.

Montgomery, W. D. (1972), "Markets in Licenses and Efficient Pollution Control Programs," Journal of Economic Theory 5.

Roberts, J. and Postlewaite, A. (1976), "The Incentives for Price-taking Behavior in Large Exchange Economies," Econometrica 44: 115-128.

Smith, V. L. (1980), "Experiments with a Decentralized Mechanism for Public Good Decisions," The American Economic Review, 70(4): 584-599.