Sharing a river

Stefan Ambec and Yves Sprumont*

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Abstract

A group of agents located along a river have quasi-linear preferences over water and money. We ask how the water should be allocated and what money transfers should be performed. We are interested in efficiency, stability (in the sense of the core), and fairness (in a sense to be defined). We first show that the cooperative game associated with that problem is convex: its core is therefore large and easily described. Next, we propose the following fairness requirement: no group of agents should enjoy a welfare higher than what it could achieve in the absence of the remaining agents. We prove that only one welfare distribution in the core satisfies this condition: its marginal contribution vector corresponding to the ordering of the agents along the river. We discuss how it could be decentralized or implemented.

Key words: common property resources, fair allocation, core.

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*Stefan Ambec: CSEF, University of Salerno, Italy. Yves Sprumont: C.R.D.E. and Département de sciences économiques, University of Montreal, Canada. We thank Eric Edmonds and Michel Truchon for very useful comments. Ambec's research was supported by C.R.D.E., CIRANO and GREEN. Sprumont acknowledges support from the Social Sciences and Humanities Research Council of Canada.

1 Introduction

Water is essential to life. Man consumes it for a variety of purposes — from drinking, cooking and washing to agricultural and industrial uses. Due to population growth and industrialization, demand has tremendously increased. On most of the earth's surface, water has now become a locally scarce resource. We are interested here in the problem of sharing *river* water. This is a challenge of considerable practical importance. Two hundred river basins in the world are shared: 148 by two countries, 30 by three, 9 by four, and 13 by five or more countries (Barret, 1994). While many countries do coordinate their consumptions (Egypt and Sudan, for instance, signed the Nile treaty, while Mali, Mauritania and Senegal founded the "Organisation pour la Mise en Valeur du fleuve Sénégal"), international disputes do occur. Two important principles are advocated in such disputes (see, e.g., Kilgour and Dinar, 1996). The *theory of unlimited territorial integrity* forbids a country to alter the natural conditions on its own territory to the disadvantage of a neighboring country. It was put forward by Egypt to claim the right to the continued and uninterrupted flow of water from the Nile river. The *theory of absolute territorial sovereignty*[1], on the other hand, states that a country has absolute sovereignty over the area of any river basin on its territory: it may freely dispose of the water flowing within its borders but cannot claim the continued and uninterrupted flow from upper basin countries. In response to the Nile treaty between Egypt and Sudan, Ethiopia invoked this doctrine to claim the right to exploit the Nile waters originating on its territory (Godana, 1985). The two doctrines are in obvious conflict, which illustrates well the tensions at work when sharing a river.

Sharing a resource over which property rights are not well defined is notoriously problematic. The economic theory literature has stressed that decentralized noncooperative behavior typically leads to inefficiency. In particular, if agents have free access to the resource and if their marginal net benefits are decreasing and eventually negative, the resource is overexploited in equilibrium: this is the famed "tragedy of the commons" (see Hardin, 1968 and Ostrom, 1990 for many examples). But sharing river water involves a twist due to the fact

[1] Also called the Harmon doctrine because it was first authoritatively stated by Judson Harmon, attorney-general of the United States, in a declaration made in 1895 concerning the Rio Grande.

that agents have unequal access to the resource. It is that twist that interests us.

Consider the case where only two agents (who could be countries) are located along a river with no tributaries; their preferences over water and money are quasi-linear, and marginal net benefits from water consumption are decreasing but always positive. To keep our point as simple as possible, suppose also that water is a completely rival good. In equilibrium, the upstream agent leaves nothing for the consumption of the other: this, of course, is typically inefficient. In a model with more agents and possible tributaries, and as long as water is not a fully nonrival good, agents located upstream still have a tendency to overconsume. In order to maximize social welfare, i.e., the sum of all agents' benefits, it is often necessary that upstream agents limit their own consumption so as to increase that of downstream agents whose marginal benefits are higher. Clearly, inducing the upstream agents to do so requires some compensatory payments.

But exactly what payments? While choosing a water consumption plan determines the level of social welfare, choosing the compensatory payments determines the distribution of that welfare. In abstract terms, therefore, the central difficulty is to agree on a welfare distribution. (Alternatively, if allocating property rights over the water may lead to efficiency, the agents still have to agree on how to distribute these rights: see Section 6). Our purpose is to propose simple principles to do so. By contrast, the policy-oriented literature on river water allocation, which is enormous (see, e.g., the numerous references in Dinar et al., 1997), is primarily concerned with the problem of designing suitable institutions or mechanisms for sharing water. This is undoubtedly of crucial importance, but it should be kept in mind that different mechanisms generally lead to different welfare distributions. We therefore believe that elementary principles for comparing such distributions are essential guidelines to evaluate and recommend particular institutional arrangements.

In a nutshell, we contend that a sustainable welfare distribution should be stable and fair. *Stability* is understood here in the sense of the core. The location of an agent along the river determines the quantity of water he controls *de facto* and, thereby, the welfare he can secure to himself. In the two-agent case with no tributaries, for instance, the upstream agent can secure to himself the benefit of consuming the total flow while the downstream agent can secure nothing. Likewise, the secure welfare of an arbitrary coalition of agents is the highest welfare it can achieve by allocating to its members the water that they control. A welfare distribution in the core gives to each coalition at least its secure welfare. We emphasize that these core constraints stem from the effective power structure, not from a legal one: agents have no obvious property rights on the river.

Fairness is admittedly a delicate and complex issue. In this paper, we suggest one, fairly minimal, criterion. Every agent, in the absence of the others, would be able to consume the full stream of water running through his location, thereby achieving what we call his *aspiration welfare* level. Sharing the river involves negative externalities in the precise sense that it is impossible to guarantee to every agent his aspiration level. We therefore suggest that everyone should take up a share of these externalities; certainly no one should end up above his aspiration welfare. Pushing the argument one step further, we define the aspiration welfare level of an arbitrary coalition of agents to be the highest welfare it could achieve in the absence of the others. Our fairness criterion requires that no coalition should enjoy more than its aspiration welfare.

The main purpose of the paper is to show that the core stability constraints (which we view as inescapable) and the upper bounds defined by the coalitional aspiration levels (which we regard as minimal) yield a unique welfare distribution. According to this distribution, an agent's welfare is just his marginal contribution to the coalition composed of his predecessors along the river. We briefly discuss how this welfare distribution could be implemented.

2 A formal statement of the problem

A river flows through a number of countries, regions or cities, henceforth called *agents*, whose set is denoted by $N = \{1, ..., n\}$. We identify agents with their location along the river and number them from upstream to downstream: $i < j$ means that i is upstream from j. We assume that different agents are located at different points along the river. If $S, T \subset N$, we write $S < T$ if $i < j$ for every $i \in S$ and every $j \in T$. We denote by $\min S$ and $\max S$, respectively, the smallest and largest members of S. It will be convenient to define the sets of *predecessors* and *followers* of agent i, respectively, by $Pi = \{j \in N : j \leq i\}$ and $Fi = \{j \in N : i \leq j\}$, and the sets of his strict predecessors and followers by $P^0 i = Pi \backslash \{i\}$ and $F^0 i = Fi \backslash \{i\}$. We often write i instead of $\{i\}$.

The river picks up volume along its course: the flow at its source, $e_1 > 0$, is increased by the amount $e_i \geq 0$ between locations $i-1$ and i, say, without loss of generality, at i. A schematic representation is given in Figure 1.

A (possibly composite) perfectly divisible good, that will be called money, is available in unbounded quantities to perform side-payments.. Agents value money and the water from the river. Agent i's utility from consuming x_i units of water and receiving a net money transfer t_i is $u_i(x_i, t_i) = b_i(x_i) + t_i$. We call $b_i : R_+ \rightarrow\ R$ agent i's *benefit function.* It is assumed to be differentiable at every $x_i > 0$, strictly increasing, and strictly concave. We denote its derivative by b_i' and assume that $b_i'(x_i)$ goes to infinity as x_i tends to 0. As a normalization, we assume $b_i(0) = 0$. The list (N, e, b), where $e = (e_1, ..., e_n)$ and $b = (b_1, ..., b_n)$, constitutes our *problem.*

A *consumption plan (for* N) is any vector $x \in R_+^N$. For an arbitrary nonempty *coalition* $S \subset N$, $x_S \in R_+^S$ denotes the restriction of x to S : it is a consumption plan for S. An *allocation* is a vector $(x, t) = (x_1, ..., x_n, t_1, ..., t_n) \in R_+^N \times R^N$ satisfying the feasibility constraints

$$\sum_{i \in N} t_i \leq 0, \tag{1}$$

$$\sum_{i \in Pj} (x_i - e_i) \leq 0 \text{ for every } j \in N. \tag{2}$$

It is important to note that the water stream e_i can only be consumed by the followers of i. This makes our problem different from that of allocating a stock of some standard private good with the possibility of side-payments.

A *welfare distribution* is any vector $z = (z_1, ..., z_n) \in R^N$ which is the utility image of some allocation (x, t) in the sense that $z_i = b_i(x_i) + t_i$ for each agent i.

Two important features of the model must be stressed. First, as is clear from the feasibility constraints (2.2), water is considered here to be a pure private good. In fact, it is to a large extent a nonrival good. Many forms of consumption by an agent do not destroy the water and leave its flow, or at least part of it, available for the consumption of downstream agents:

see Young and Haveman (1985) for a discussion. If water was a pure nonrival good, however, there would be no consumption sharing problem at all. We therefore focus exclusively on the rival forms of consumption. This is not to say that the partially nonrival nature of water does not raise problems, but we ignore them here.

The second feature is that the marginal costs of consumption never exceed the marginal benefits, as reflected by the assumption that the b_i functions are increasing over the whole nonnegative real line. This assumption is not standard in the literature on common property resources: the reason, of course, is that the tragedy of the commons arises only if marginal costs eventually do exceed marginal benefits. In our model, however, noncooperative behavior is inefficient *even* when net benefits are always increasing; moreover, the assumption allows us to focus on the type of inefficiency that is particular to sharing river water, namely, the tendency for upstream agents to overconsume relative to downstream agents.

3 Efficiency

Because preferences are quasi-linear, an allocation $(x^*(N), t^*(N))$ is (Pareto) efficient if and only if it maximizes the sum of all agents' benefits and wastes no money. We call $x^*(N)$ an *optimal consumption plan.*

This section describes the structure of an optimal consumption plan. In fact, we consider consumption plans maximizing the total benefit of an arbitrary coalition $S \subset N$ under a more general set of constraints than just the corresponding feasibility conditions. We state a number of lemmata concerning such plans, and how they are affected by changes in the constraints. These lemmata, which are proved in Appendix 1, will be used repeatedly in Sections 4 and 5.

Let $\emptyset \neq S \subset N$ and let $T \subset S$. Throughout Section 3 and Appendix 1, T and S are fixed and all agents under consideration are understood to be members of S. We therefore often simplify notations by dropping reference to S; for instance, $i \leq j$ means that $i \in Pj \cap S$, and $\sum_i x_i$ means $\sum_{i \in S} x_i$.

Notation. Fix two numbers α, ω that are *admissible* in the sense that $0 \leq \alpha$ and $0 \leq$

$\omega \leq \sum_i e_i + \alpha$. Fix also a consumption plan $\overline{x}_T$ that is *feasible for T in S given α, ω)* in the sense that there exists a consumption plan x for S satisfying the constraints

$$x_T = \overline{x}_T \tag{3}$$

and

$$\sum_{i \leq j} (x_i - e_i) \leq \alpha - \theta_j \omega \tag{4}$$

for every $j \in S$, where $\theta_j = 1$ if $j = \max S$ and $\theta_j = 0$ otherwise. Denote by $x^*(S; \alpha, \omega, \overline{x}_T)$ any consumption plan for S that maximizes $\sum_i b_i(x_i)$ subject to those constraints. Extending our terminology, we call such a plan *optimal*. If $T = \emptyset$, we alleviate notations and write $x^*(S; \alpha, \omega)$ instead of $x^*(S; \alpha, \omega, \overline{x}_\emptyset)$. If $\alpha = \omega = 0$, we write $x^*(S; \overline{x}_T)$ instead of $x^*(S; 0, 0, \overline{x}_T)$ and $x^*(S)$ instead of $x^*(S; 0, 0, \overline{x}_\emptyset)$.

The parameter α may be viewed as an extra flow made available at $\min S$ for possible consumption by any member of S while ω is interpreted as a minimal flow that must be left over by S at $\max S$. For any admissible α, ω and any feasible $\overline{x}_T$, our assumptions guarantee that the optimal plan $x^*(S; \alpha, \omega, \overline{x}_T)$ is unique. It is described in Lemma 1 below.

Lemma 1. *If $\sum_{i \leq j} (x_i^*(S; \alpha, \omega, \overline{x}_T) - e_i) = \alpha - \theta_j \omega$ for some j , let S^* be the set of predecessors in S of the largest such j ; set $S^* = \emptyset$ otherwise. Then, i) $S \backslash T \subset S^*$ and ii) if $S^* \neq \emptyset$, there is a partition $\{S_k^*\}_{k=i,\ldots,K}$ of S^* and a list $\{\beta_k\}_{k=1,\ldots,K}$ of positive numbers such that*

$$S_k^* < S_{k'}^* \quad \textit{and } \beta_k \geq \beta_{k'} \quad \textit{whenever } k < k', \tag{5}$$

$$b_i'(x_i^*(S; \alpha, \omega, \overline{x}_T)) = \beta_k \quad \textit{for every } i \in S_k^* \backslash T \quad \textit{and every } k = 1, \ldots K, \quad \textit{and} \tag{6}$$

$$\sum_{i \in S_k^*} (x_i^*(S; \alpha, \omega, \overline{x}_T) - e_i) = \alpha - \theta_{\max S_k^*} \omega \quad \textit{for every } k = 1, \ldots, K. \tag{7}$$

According to Lemma 1, the marginal benefits of the agents in $S \backslash T$ decrease weakly as we move downstream. Moreover, if two agents in $S \backslash T$, say, $j < j''$, have different marginal benefits, some constraint must be binding between them: there exists $j' \in S$, $j \leq j' < j''$, such that $\sum_{i \leq j'} (x_i - e_i) = \alpha$.

In our next three lemmata, we formulate useful monotonicity — or solidarity— properties of optimal plans. We use the vector inequality notation $\leq, <, \ll$. The first two results assert that the consumption of no member of S at an optimal plan can be reduced if more water is made available at $\min S$ or if less water must be left over at $\max S$.

Lemma 2. *For any admissible* $\alpha, \omega,$*and any feasible* $\overline{x}_T$, $\alpha \leq \alpha' \Rightarrow x^*(S;\alpha,\omega,\overline{x}_T) \leq x^*(S;\alpha',\omega,\overline{x}_T)$.

Lemma 3. *For any admissible* α, ω', *and any feasible* $\overline{x}_T$, $0 \leq \omega \leq \omega' \Rightarrow x^*(S;\alpha,\omega,\overline{x}_T) \geq x^*(S;\alpha,\omega',\overline{x}_T)$.

These lemmata can be used to show that decreasing the consumption of the members of T cannot reduce the consumption of anyone in $S\backslash T$. This is the content of our next lemma.

Lemma 4. *Suppose* $T \neq S$. *For any admissible* α, ω *and any feasible* $\overline{x}_T, 0 \leq \overline{y}_T \leq \overline{x}_T \Rightarrow x^*_{S\backslash T}(S;\alpha,\omega,\overline{y}_T) \geq x^*_{S\backslash T}(S;\alpha,\omega,\overline{x}_T)$.

With these results in hand, we are now ready to analyze the cooperative game generated by our problem.

4 Core stability: lower bounds on welfare

Following Greenberg and Weber (1986), we call a coalition S *consecutive* if $k \in S$ whenever $i, j \in S$ and $i < k < j$. If S is consecutive, we call

$$v(S) = \sum_{i \in S} b_i(x^*_i(S)) \tag{8}$$

the *secure benefit* of S. Now, every coalition T admits a unique coarsest partition into consecutive components: denote it T. The *secure benefit* of T obtains by summing up the secure benefits of its consecutive components,

$$v(T) = \sum_{S \in T} v(S), \tag{9}$$

where $v(S)$ is given by (4.1). Coalition T cannot secure more than (4.2) because any water left over by one of its connected components cannot be guaranteed for the consumption of any other component. We say that v is the *game generated by the problem* (N, e, b). It is an example of what Greenberg and Weber (1986, 1993) call a "consecutive" game: one in which only consecutive coalitions generate surplus. We insist that no property rights exist in the problem (N, e, b) and that the game v is an expression of the natural, effective —but not legal— distribution of power among agents.

A (welfare) distribution $z = (z_1, ..., z_n)$ is a *core distribution* if $\sum_{i \in S} z_i \geq v(S)$ for every $S \subset N$. An allocation that does not generate a core distribution would be unstable: some coalition could object to it on the basis that it can secure on its own a higher welfare to all its members. Fortunately, core distributions do exist in the present context. This is because the game v generated by our problem is *convex* in the sense of Shapley (1971), i.e., $v(S) - v(S \backslash i) \leq v(T) - v(T \backslash i)$ whenever $i \in S \subset T \subset N$. Informally, a convex game is one where cooperation exhibits increasing marginal returns. To prove convexity of v, the following lemma, whose straightforward proof is in Appendix 2, will be useful.

Lemma 5. *If S, T are two coalitions such that $S < T$, then i) $x^*(T) \leq x^*_T(S \cup T)$ and ii) $x^*_S(S \cup T) \leq x^*(S)$.*

We are now ready to establish our claim. To put it in perspective, it might be useful to recall that an arbitrary consecutive game is not necessarily convex.

Proposition. *The game v generated by the problem (N, e, b) is convex.*

Proof. Fix $i \in S \subset T \subset N$. Let R be the (unique, consecutive) coalition in the partition S containing i and let Q be the unique coalition in T containing i. Note that $R \subset Q$. Given (4.2), we need only check that

$$v(R) - v(R \backslash i) \leq v(Q) - v(Q \backslash i). \tag{10}$$

Let $R_P := R \cap P^0 i$, $R_F := R \cap F^0 i$, and define Q_P and Q_F similarly. Note that $v(R \backslash i) = v(R_P) + v(R_F)$ and $v(Q \backslash i) = v(Q_P) + v(Q_F)$. Moreover, $R_P \subset Q_P$, $R_F \subset Q_F$, and R, R_P, R_F as well as Q, Q_P, Q_F are all consecutive.

Step 1. We claim that

$$v(R_P \cup i) - v(R_P) \leq v(Q_P \cup i) - v(Q_P), \tag{11}$$

$$v(R_F \cup i) - v(R_F) \leq v(Q_F \cup i) - v(Q_F). \tag{12}$$

To prove (4.4), let $d_j = x_j^*(R_P) - x_j^*(R_P \cup i)$ for each $j \in R_P$. This quantity is nonnegative because of Lemma 5. By definition,

$$v(R_P \cup i) - v(R_P) = \sum_{j \in R_P} [b_j(x_j^*(R_P) - d_j) - b_j(x_j^*(R_P))] + b_i(e_i + \sum_{j \in R_P} d_j). \tag{13}$$

Next, define the following consumption plan for the consecutive coalition $Q_P \cup i$:

$$x_j^0 = \left\{ \begin{array}{cc} x_j^*(Q_P) & \text{if } j \in Q_P \backslash R_P, \\ x_j^*(Q_P) - d_j & \text{if } j \in R_P, \\ e_i + \sum_{j \in R_P} d_j & \text{if } j = i. \end{array} \right\}$$

By definition of the increments d_j, this consumption plan meets the same feasibility constraints as $x^*(Q_P \cup i)$; note in particular that $x_j^0 \geq 0$ for every $j \in R_P$ because Lemma 5 guarantees that $x_j^*(Q_P) \geq x_j^*(R_P)$. Therefore,

$$v(Q_P \cup i) - v(Q_P) \geq \sum_{j \in R_P} [b_j(x_j^*(Q_P) - d_j) - b_j(x_j^*(Q_P))] + b_i(e_i + \sum_{j \in R_P} d_j). \tag{14}$$

Moreover, by concavity of the benefit functions, $b_j(x_j^*(R_P) - d_j) - b_j(x_j^*(R_P)) \leq b_j(x_j^*(Q_P) - d_j) - b_j(x_j^*(Q_P)) \leq 0$ for every $j \in R_P$. Taking these inequalities into account, (4.6) and (4.7) imply (4.4). The same argument, *mutatis mutandis*, establishes (4.5).

Step 2. By repeated application of (4.4) and (4.5), we obtain $v(R_P \cup Q_F \cup i) - v(R_P \cup R_F \cup i) \geq v(Q_F \cup i) - v(R_F \cup i)$ and $v(Q_P \cup Q_F \cup i) - v(R_P \cup Q_F \cup i) \geq v(Q_P \cup i) - v(R_P \cup i)$. Therefore,

$$\begin{aligned} & v(Q) - v(R) \\ = \; & [v(Q_P \cup Q_F \cup i) - v(R_P \cup Q_F \cup i)] + [v(R_P \cup Q_F \cup i) - v(R_P \cup R_F \cup i)] \\ \geq \; & [v(Q_P \cup i) - v(R_P \cup i)] + [v(Q_F \cup i) - v(R_F \cup i)] \\ \geq \; & v(Q_P) - v(R_P) + v(Q_F) - v(R_F) \\ = \; & v(Q \backslash i) - v(R \backslash i), \end{aligned}$$

where the second inequality holds again because of (4.4) and (4.5). We are done. □

The above proposition is important because the core of a convex game has a simple and well-known structure. If π is a bijection from N to N, the so-called *marginal contribution vector* z^π is the welfare distribution assigning to each agent his marginal contribution to the coalition made up of his strict predecessors in the ordering generated by π, i.e., $z_i^\pi = v(\{j : \pi(j) \leq \pi(i)\}) - v(\{j : \pi(j) < \pi(i)\})$ for every $i \in N$. Shapley (1971) has shown that the core of a convex game is the convex hull of all the marginal contribution vectors.

Two particular welfare distributions occupying distinguished positions in the core of a convex game are the *Shapley value* and the Dutta-Ray (1989) *constrained egalitarian welfare distribution.* The former is the barycenter of the core; the latter is the unique maximal element of the Lorenz partial order in the core. While these two welfare distributions are certainly of interest in our problem, we emphasize that they are defined from the sole knowledge of the game v. This, we believe, is a shortcoming. If the game v generated by the problem (N, e, b) is all that counts from the stability viewpoint of the core, it does not contain all the information that is relevant to a complete analysis. We contend, in particular, that the problem (N, e, b) generates upper bounds on welfare that are very appealing from the viewpoint of fairness.

5 Fairness: upper bounds on welfare

In the absence of the other agents, agent i would be able to consume the full stream of water running through his location, thereby enjoying his *aspiration welfare*

$$w(i) = b_i(\sum_{j \in Pi} e_j).$$

Of course, the welfare distribution $(w(1), ..., w(n))$ is not feasible: $\sum_{i \in N} w(i) > v(N)$ as soon as N contains at least two agents. In Moulin's (1990a) terms, the problem (N, e, b) exhibits *negative group externalities.* In such a context, it is natural to ask that everyone takes up a share of these externalities; certainly no one should end up above his aspiration welfare.

This argument generalizes to coalitions in a very natural way. The *aspiration welfare* of an arbitrary coalition S is the highest welfare it could achieve in the absence of $N \backslash S$. It is obtained by choosing a consumption plan x_S maximizing $\sum_{i\in S} b_i(x_i)$ subject to the constraints

$$\sum_{j\in Pi\cap S} x_j \leq \sum_{j\in Pi} e_j \text{ for every } i\in S. \tag{15}$$

This problem has a unique solution, which we denote by $x^{**}(S)$. The aspiration welfare of S is thus

$$w(S)=\sum_{i\in S} b_i(x_i^{**}(S))$$

and we say that a welfare distribution z *satisfies the aspiration upper bounds* if $\sum_{i\in S} z_i \leq w(S)$ for every $S\subset N$.

Combining these fairness upper bounds with the core lower bounds yields a remarkable result. It turns out that only one welfare distribution passes both tests: it is the *downstream incremental* distribution z^* defined by $z_i^* = v(Pi) - v(P^0 i)$ for each $i\in N$. Notice that since the game v is convex, z^* admits a simple characterization: among all core distributions, it is the one that lexicographically maximizes the welfare of agents $n, n-1, ..., 2, 1$.

Theorem. *The downstream incremental distribution z^* is the unique core distribution satisfying the aspiration upper bounds.*

We will use an additional piece of notation and a lemma. If $\emptyset \neq S\subset N$, $T\subset S$, and $\overline{x}_T$ is feasible for T in S, define $v(S;\overline{x}_T)=\sum_{i\in S} b_i(x_i^*(S;\overline{x}_T))$, where we recall that $x^*(S;\overline{x}_T)$ maximizes the total benefit to S subject to allocating $\overline{x}_T$ to T.

Lemma 6. *Let $s, q\in N$, $s<q$, and define $S:=Ps$, $Q:=Pq$. Let $\emptyset\neq T\subset S$, and let $\overline{x}_T$ be feasible for T in S. Then, $0\leq \overline{y}_T \leq \overline{x}_T \Rightarrow v(S;\overline{y}_T) - v(S;\overline{x}_T) \leq v(Q;\overline{y}_T) - v(Q;\overline{x}_T)$.*

The proof of this fact may be found in Appendix 2. We are now ready to establish our theorem.

Proof of the Theorem. The argument is divided into three steps.

Step 1: The downstream incremental distribution z^* is a core distribution.

The distribution z^* is just the marginal contribution vector corresponding to the ordering $1, ..., n$. That vector is a core distribution because v is convex, as asserted by the Proposition in Section 4.

Step 2: If a core distribution z satisfies the aspiration upper bounds, then $z = z^*$.

Key to the proof is the straightforward observation that $v(Pi) = w(Pi)$ for every $i \in N$. Since this is true for $i = 1$, the core inequalities and the aspiration upper bounds immediately imply that $z_1 = z_1^*$. Next, proceed inductively. Fix $j < n$ and suppose $z_i = z_i^*$ for all $i \leq j$. Since $v(P(j+1)) = w(P(j+1))$, the core constraints and the aspiration upper bounds force $\sum_{i \in P(j+1)} z_i = v(P(j+1))$, hence $z_{j+1} = v(P(j+1)) - \sum_{i \in Pj} z_i$. By the induction hypothesis, $\sum_{i \in Pj} z_i = \sum_{i \in Pj} z_i^* = v(Pj)$. Therefore, $z_{j+1} = v(P(j+1)) - v(Pj) = z_{j+1}^*$, as desired.

Step 3: z^* satisfies the aspiration upper bounds.

Fix an arbitrary coalition S, write PS for $P \max S$, and compute

$$\begin{aligned}
\sum_{j \in S} z_j^* &= \sum_{j \in S} [v(Pj) - v(P^0 j)] \\
&= \sum_{j \in S} [v(Pj; x^*_{Pj \setminus S}(Pj)) - v(P^0 j)] \\
&\leq \sum_{j \in S} [v(Pj; x^*_{Pj \setminus S}(Pj)) - v(P^0 j; x^*_{Pj \setminus S}(Pj))] \\
&\leq \sum_{j \in S} [v(Pj; x^*_{Pj \setminus S}(PS)) - v(P^0 j; x^*_{Pj \setminus S}(PS))].
\end{aligned}$$

The last inequality holds by Lemma 6 because Lemma 5 guarantees that $x^*_{Pj \setminus S}(PS) \leq x^*_{Pj \setminus S}(Pj)$ for every $j \in S$. Writing $\tilde{x}(Pj) := x^*(Pj; x^*_{Pj \setminus S}(PS))$ and $\tilde{x}(P^0 j) := x^*(P^0 j; x^*_{Pj \setminus S}(PS))$ whenever $j \in S$, we obtain

$$\begin{aligned}
\sum_{j \in S} z_j^* &\leq \sum_{j \in S} [\sum_{i \in P^0 j \cap S} [b_i(\tilde{x}_i(Pj)) - b_i(\tilde{x}_i(P^0 j))] + b_j(\tilde{x}_j(Pj))] \\
&= \sum_{i \in S} [\sum_{j \in F^0 i \cap S} [b_i(\tilde{x}_i(Pj)) - b_i(\tilde{x}_i(P^0 j))] + b_i(\tilde{x}_i(Pi))].
\end{aligned}$$

For every $i \in S$, every $j \in F^0 i \cap S$, and every $k \in F^0 j \cap S$, Lemma 5 guarantees that $\tilde{x}_i(P^0 k) \leq \tilde{x}_i(Pj) \leq \tilde{x}_i(P^0 j) \leq \tilde{x}_i(Pi)$. Therefore, $\sum_{j \in F^0 i \cap S} [\tilde{x}_i(Pj) - \tilde{x}_i(P^0 j)] + \tilde{x}_i(Pi) \geq 0$. By concavity of b_i,

$$\begin{aligned}
&\sum_{j \in F^0 i \cap S} [b_i(\tilde{x}_i(Pj)) - b_i(\tilde{x}_i(P^0 j))] + b_i(\tilde{x}_i(Pi)) \\
\leq\ & b_i(\sum_{j \in F^0 i \cap S} [\tilde{x}_i(Pj) - \tilde{x}_i(P^0 j)] + \tilde{x}_i(Pi)).
\end{aligned}$$

Writing

$$y_i := \sum_{j \in F^0 i \cap S} [\tilde{x}_i(Pj) - \tilde{x}_i(P^0 j)] + \tilde{x}_i(Pi),$$

we therefore get

$$\sum_{j\in S} z_j^* \leq \sum_{i\in S} b_i(y_i).$$

We complete the proof by showing that $\sum_{i\in S} b_i(y_i) \leq w(S)$. To do so, it is certainly enough to show that

$$\sum_{i\in Pj\cap S} (y_i - e_i) \leq 0 \text{ for every } j \in S. \tag{16}$$

Fix $j \in S$. Compute

$$\begin{aligned}
\sum_{i\in Pj\cap S} y_i &= \sum_{i\in Pj\cap S} \sum_{k\in F^0 i\cap Pj\cap S} [\tilde{x}_i(Pk) - \tilde{x}_i(P^0k)] \\
&\quad + \sum_{i\in Pj\cap S} \sum_{k\in F^0 j\cap S} [\tilde{x}_i(Pk) - \tilde{x}_i(P^0k)] + \sum_{i\in Pj\cap S} \tilde{x}_i(Pi) \\
&= \sum_{k\in Pj\cap S} \sum_{i\in P^0k\cap S} [\tilde{x}_i(Pk) - \tilde{x}_i(P^0k)] \\
&\quad + \sum_{i\in Pj\cap S} \sum_{k\in F^0 j\cap S} [\tilde{x}_i(Pk) - \tilde{x}_i(P^0k)] + \sum_{i\in Pj\cap S} \tilde{x}_i(Pi),
\end{aligned}$$

so that finally

$$\begin{aligned}
\sum_{i\in Pj\cap S} y_i &= \sum_{k\in Pj\cap S} [\sum_{i\in Pk\cap S} \tilde{x}_i(Pk) - \sum_{i\in P^0k\cap S} \tilde{x}_i(P^0k)] \\
&\quad + \sum_{i\in Pj\cap S} \sum_{k\in F^0 j\cap S} [\tilde{x}_i(Pk) - \tilde{x}_i(P^0k)].
\end{aligned} \tag{17}$$

For every $k \in S$, we have by definition of $\tilde{x}(Pk)$ and $\tilde{x}(P^0k)$,

$$\begin{aligned}
\sum_{i\in Pk\cap S} \tilde{x}_i(Pk) + \sum_{i\in Pk\setminus S} x_i^*(PS) &= \sum_{i\in Pk} e_i, \\
\sum_{i\in P^0k\cap S} \tilde{x}_i(P^0k) + \sum_{i\in P^0k\setminus S} x_i^*(PS) &= \sum_{i\in P^0k} e_i.
\end{aligned}$$

Subtracting the second of these equations from the first and replacing in (5.3) yields

$$\sum_{i\in Pj\cap S} y_i = \sum_{k\in Pj\cap S} e_k + \sum_{i\in Pj\cap S} \sum_{k\in F^0 j\cap S} [\tilde{x}_i(Pk) - \tilde{x}_i(P^0k)] \leq \sum_{k\in Pj\cap S} e_k,$$

where the inequality holds because $\tilde{x}_i(Pk) \leq \tilde{x}_i(P^0k)$ whenever $i \in Pj \cap S$ and $k \in F^0 j \cap S$ because of Lemma 5. Since j was arbitrary, this establishes (5.2) and finishes the proof. □

6 Decentralization and implementation

We argued so far that an optimal water consumption plan must be accompanied by appropriate side-payments in order to lead to a sustainable welfare distribution. We further suggested that a sustainable welfare distribution should be stable and fair. We showed how the core stability constraints and the aspiration upper bounds pin down the so-called downstream incremental distribution. That approach is quite abstract and very much cooperative in spirit: it seeks to determine what general agreements have a better chance to be acceptable to all parties.

The focus of the current section, by contrast, is procedural and noncooperative. We briefly discuss various institutional arrangements or mechanisms in which the decentralized behavior of the concerned agents could lead to the downstream incremental welfare distribution. We distinguish two forms of decentralized behavior: myopic competitive behavior is assumed in the first subsection, sophisticated strategic behavior in the second. As most of the discussion below involves familiar concepts, we will deliberately keep the presentation somewhat informal. We refer the reader to Moore (1992) for an introduction to the theory of implementation under complete information.

6.1 Decentralization by markets

A simple procedure to avoid the inefficient use of river waters consists in assigning explicit property rights to the concerned agents and set up markets where they can exchange these rights. This is done in practice: in the irrigation service area of Alicante in Spain, for instance, agents are endowed with volumetric water rights from specific sources which they may then exchange in a public auction held every Sunday morning. Trade is enforced by an executive commission elected by the members (Ostrom, 1990, Reidinger, 1994).

To formalize such a procedure in the context of our model, define an array $\Theta = (\theta_{ij})_{i,j \in N}$, where θ_{ij} is the proportion of tributary j's water owned by agent i: thus, $0 \leq \theta_{ij} \leq 1$ for all i, j and $\sum_{i \in N} \theta_{ij} = 1$ for every j. The list (N, e, b, Θ) generates an $(n+1)$-good exchange economy in the following way. Water from each tributary j is considered as a separate good; agent i's endowment in that good is $\theta_{ij}e_j$ and endowments in money are arbitrary. If x_{ij} denotes i's consumption of water from tributary j, $\mathbf{x}_i := (x_{ij})_{j\in N}$, and t_i is the net money transfer received, agent i's preferences are represented by the utility function $U_i(\mathbf{x}_i, t_i) = b_i(\sum_{j\in Pi} x_{ij}) + t_i$. An allocation for that exchange economy is a list $(\mathbf{x}, t)$, where $\mathbf{x} = (x_{ij})_{i,j\in N}$ and $t = (t_i)_{i\in N}$ satisfy the constraints that $\sum_{i\in N} x_{ij} \leq e_j$ for every $j \in N$ and $\sum_{i\in N} t_i \leq 0$.

A (competitive) equilibrium for (N, e, b, Θ) is defined in the obvious way. Normalizing money endowments to zero, it is straightforward to check that i's utility at equilibrium is

$$z_i(N, e, b, \Theta) = b_i(x_i^*) - b_i'(x_i^*)x_i^* + \sum_{j\in N} b_j'(x_j^*)\theta_{ij}e_j,$$

where x^* is the optimal water consumption plan in the problem (N, e, b) discussed in Section 3. We may now search for an array of property rights Θ that would generate the downstream incremental welfare distribution. Writing the latter $z^*(N, e, b)$ to emphasize its dependence upon the problem at hand, we need to solve the system

$$\begin{aligned} z_i(N, e, b, \Theta) &= z_i^*(N, e, b) \qquad \text{for all } i \in N, \\ \sum_{i\in N} \theta_{ij} &= 1 \qquad \text{for all } j \in N. \end{aligned}$$

This is a linear system in Θ. While it may have several solutions, the important observation is that they will typically change with the preference profile b: the simplest way to see this is to note that $z_1^*(N, e, b) = b_1(e_1)$ does not depend on $(b_j)_{j\neq 1}$ while $z_1(N, e, b, \Theta)$ generally does. It is therefore necessary to know the agents' preferences in order to design property rights that would generate the downstream incremental welfare distribution through competitive exchange. In particular, endowing agents with equal property rights, a solution that is central in the literature on the classical fair-division problem (Varian, 1974, Champsaur and Laroque, 1981) and important in the standard formulation of the tragedy of the commons (Moulin, 1990b), would not always lead to the downstream incremental distribution. It is easy to see that the welfare distribution at the competitive equilibrium from equal endowments may in fact violate the core constraints of Section 4 as well as the aspiration upper bounds of Section 5.

6.2 Implementation through game forms

When the number of agents is small, competitive behavior is unlikely. Implementation in the game-theoretic sense becomes important. To coordinate international river management, countries often join institutions or sign treaties that specify negotiation rules on various matters. For instance, the "principe d'approbation des Etats" included in the treaty founding the "Organisation pour la Mise en Valeur du fleuve Sénégal" states that a country cannot change the water flow without the consent of all members (Ambec, 1997). Such rules sometimes come close to explicit game forms.

It is easy to implement in subgame perfect equilibrium the downstream incremental welfare distribution — or, more precisely, the rule that associates with each conceivable preference profile b the corresponding downstream incremental distribution $z^*(N, e, b)$. As we already pointed out, that distribution lexicographically maximizes the welfare of agents $n, n-1, ..., 2, 1$ subject to the core constraints. This suggests an extensive game form in which $n, n-1, ..., 2, 1$ are successively allowed to make offers. If $s \in N$, call an *allocation for* Ps any vector $(x, t) \in R_+^{Ps} \times R^{Ps}$ such that $\sum_{i \in Ps} t_i \leq 0$ and $\sum_{i \in Pj}(x_i - e_i) \leq 0$ for every $j \in Ps$. In the first stage, agent n proposes an allocation for $Pn = N$. Agents $n-1, ..., 1$ are successively asked whether they agree with the proposed allocation. If they all do, the allocation is enforced. Otherwise, agent n gets the bundle $(x_n, t_n) = (e_n, 0)$. Agent $n-1$ may now propose an allocation for $P(n-1)$, which in turn needs the unanimous successive approval of $n-2, ..., 1$ to be enforced; otherwise $n-1$ gets $(e_{n-1}, 0)$. If the last stage of this game form is ever reached, agent 2 proposes an allocation for $P2$ which is enforced if agent 1 agrees; otherwise, 2 gets $(e_2, 0)$ and 1 gets $(e_1, 0)$. Straightforward backwards induction shows that every subgame perfect equilibrium of the game generated by this game form and an arbitrary preference profile b yields the downstream incremental welfare distribution $z^*(N, e, b)$.

7 Concluding comments

The specificity of the problem analyzed in this paper stems from the nature of the feasibility constraints at play: agents are ordered and water can only be transferred downstream. We showed how this extremely rigid structure suggests powerful guidelines for collective choices.

Other interesting allocation problems involve highly structured feasibility constraints reminiscent of inequalities (2.2). Intertemporal models, for instance, where commodities cannot be consumed before they are produced, share some essential features with our setup. More generally, networks of exchange where not all participants can meet are of a similar nature. We believe that the theory of collective choices has much to gain from a systematic exploitation of the structure of the feasibility constraints in such environments.

Appendix 1

Proof of Lemma 1. Write $x^*(S;\alpha,\omega,\overline{x}_T)=x$. To prove i), note first that the inclusion $S\backslash T\subset S^*$ is trivial if $S^*=S$. Otherwise, let $j_1,...,j_L=\max S$ denote the members of $S\backslash S^*$, with $j_l<j_{l'}$ whenever $l<l'$. Since (3.2) is a strict inequality for $j=\max S$ and since $b_{\max S}$ is strictly increasing, it must be that $\max S\in T$. Repeating this argument, we conclude, successively, that $j_L,...,j_1$ are all members of T and, therefore, $S\backslash T\subset S^*$.

To prove ii), assume now that S^* is nonempty. Denoting by λ_j the multiplier associated with constraint (3.2) in the maximization problem defining x, the first-order conditions yield

$$b_j'(x_j)=\sum_{i\geq j}\lambda_i \text{ for all } j\in S\backslash T, \tag{18}$$

$$\lambda_j[\sum_{i\leq j}(x_i-e_i)-\alpha+\theta_j\omega]=0 \text{ for all } j\in S, \tag{19}$$

$$\lambda_j\geq 0 \text{ for all } j\in S. \tag{20}$$

Let $j_1^*,...,j_K^*$ be the agents j for which (3.2) is an equality; there is at least one since $S^*\neq\emptyset$. Define $\beta_1=\sum_{j\geq j_1^*}\lambda_j$, $S_1^*=\{j\in S:j\leq j_1\}$, and, if $1<K$, define $\beta_k=\sum_{j\geq j_k^*}\lambda_j$ and $S_k^*=\{j\in S:j_{k-1}\leq j\leq j_k\}$ whenever $1<k\leq K$. Conditions (3.5) are then satisfied. Moreover, conditions (9.1) and (9.2) imply (3.4) because $\lambda_j=0$ for every $j\neq j_1^*,...,j_K^*$. Finally, conditions (9.1) and (9.3) guarantee (3.3). □

Proof of Lemma 2. Write $x^*(S;\alpha,\omega,\overline{x}_T)=x$, $x^*(S;\alpha',\omega,\overline{x}_T)=y$, and recall that all agents under consideration belong to S. If $y=x$, we are done. Otherwise, consider all the agents $i\in S$ such that $y_i\neq x_i$. Number them $i_1,...,i_L$, with $i_l<i_{l'}$ whenever $l<l'$. By optimality of y,

$$\sum_{i\geq i_1}(y_i-x_i)\geq 0. \tag{21}$$

We claim that

$$y_{i_1}-x_{i_1}>0. \tag{22}$$

Suppose that the opposite strict inequality holds. Let j be the smallest follower of i_1 such that $y_j-x_j>0$, which exists because of (9.4). Observe that

$$\sum_{i\leq k}(y_i-e_i)<\sum_{i\leq k}(x_i-e_i)\leq\alpha\leq\alpha' \text{ whenever } i_1\leq k<j. \tag{23}$$

Define y^ε by $y^\varepsilon_{i_1}=y_{i_1}+\varepsilon$, $y^\varepsilon_j=y_j-\varepsilon$, and $y^\varepsilon_i=y_i$ for all $i\neq i_1,j$. The inequalities in (9.6) guarantee that, for sufficiently small $\varepsilon>0$, y^ε satisfies the constraints of the maximization problem defining y, namely, $\sum_{i\leq k}(y^\varepsilon_i-e_i)\leq\alpha'$ whenever $k<\max S$, $\sum_{i\leq\max S}(y^\varepsilon_i-e_i)\leq\alpha'-\omega$, and $y^\varepsilon_T=\overline{x}_T$. By optimality of x , however, $b'_{i_1}(x_{i_1})\geq b'_j(x_j)$ and, since b_{i_1} and b_j are strictly concave, $b'_{i_1}(y_{i_1})>b'_{i_1}(x_{i_1})$ and $b'_j(x_j)>b'_j(y_j)$, hence, $b'_{i_1}(y_{i_1})>b'_j(y_j)$. For ε small enough, therefore, $\sum_i[b_i(y^\varepsilon_i)-b_i(y_i)]>0$, contradicting the optimality of y. We have proved (9.5).

Suppose now, contrary to the claim, that $y_{i_l}-x_{i_l}<0$ for some $l\geq 2$. Choose l minimal. By optimality of y and strict concavity of $b_{i_{l-1}}$ and b_{i_l}, we know that $b'_{i_{l-1}}(x_{i_{l-1}})>b'_{i_l}(x_{i_l})$. By Lemma 1, there is some k, $i_{l-1}\leq k<i_l$, such that the constraint on x at k is binding, i.e., $\sum_{i\leq k}(x_i-e_i)=\alpha$. It follows that $\sum_{i>k}(x_i-e_i)=-\omega$ and therefore

$$\sum_{i\geq i_l}(y_i-x_i)\geq 0 \tag{24}$$

since $\sum_{i\geq i_l}(y_i-x_i)=\sum_{i>k}(y_i-x_i)=\sum_{i>k}(y_i-e_i)+\omega=\sum_{i\leq k}(e_i-y_i)+\alpha'\geq-\alpha'+\alpha'=0$. Now, mimicking the argument showing that (9.4) implies (9.5), we obtain from (9.7) that $y_{i_l}-x_{i_l}>0$, which is the desired contradiction. □

Proof of Lemma 3. Write now $x^*(S;\alpha,\omega,\overline{x}_T) = x$ and $x^*(S;\alpha,\omega',\overline{x}_T) = y$. If $y = x$, we are done. Otherwise, consider all agents $i \in S$ such that $y_i \neq x_i$. Number them $i_1, ..., i_L$, with $i_l < i_{l'}$ whenever $l < l'$. By optimality of x,

$$\sum_{i \leq i_L} (y_i - x_i) \leq 0. \tag{25}$$

We claim that

$$y_{i_L} - x_{i_L} < 0. \tag{26}$$

Suppose the converse strict inequality holds. Let j be the largest predecessor of i_L such that $y_j - x_j < 0$, which exists because of (9.8). Since $\sum_{i \leq j}(y_i - x_i) < 0$, we know that $\sum_{i \leq j}(y_i - e_i) < \alpha$. Letting j' be the smallest strict follower of j in $S \backslash T$, it follows that

$$\sum_{i \leq k}(y_i - e_i) < \alpha \text{ whenever } j \leq k < j'. \tag{27}$$

By optimality of x and strict concavity of b_j and $b_{j'}$ however, we know that $b'_j(y_j) > b'_{j'}(y_{j'})$, which, by Lemma 1, contradicts (9.10). We have proved (9.9).

Suppose now, contrary to the claim, that $y_{i_l} - x_{i_l} > 0$ for some $l \leq L - 1$. Choose l maximal. By optimality of y and strict concavity of b_{i_l} and $b_{i_{l+1}}$, $b'_{i_l}(x_{i_l}) > b'_{i_{l+1}}(x_{i_{l+1}})$. By Lemma 1, there is some k, $i_l \leq k < i_{l+1}$, such that $\sum_{i \leq k}(x_i - e_i) = \alpha$. It follows that

$$\sum_{i \leq i_l}(y_i - x_i) \leq 0 \tag{28}$$

since $\sum_{i \leq i_l}(y_i - x_i) = \sum_{i \leq k}(y_i - x_i) = \sum_{i \leq k}(y_i - e_i) - \sum_{i \leq k}(x_i - e_i) \leq \alpha - \alpha = 0$. Mimicking the argument showing that (9.8) implies (9.9), we obtain from (9.11) that $y_{i_l} - x_{i_l} < 0$, a contradiction. $\square$

Proof of Lemma 4. Let $T, \alpha, \omega, \overline{x}_T$ and $\overline{y}_T$ satisfy the assumptions of the lemma. Write $x^*(S;\alpha,\omega,\overline{x}_T) = x$ and $x^*(S;\alpha,\omega,\overline{y}_T) = y$. The case where $\overline{y}_T = \overline{x}_T$ being straightforward, assume that $\overline{y}_T < \overline{x}_T$ (recall our notation for vector inequalities). Since $\overline{x}_T$ may be transformed into $\overline{y}_T$ coordinate by coordinate, there is no loss of generality in assuming that the two vectors differ in only one coordinate, say, t. Since all agents are assumed to be members of S, we further abuse our notation slightly and write $P^0t = \{i \in S : i < t\}$ and $F^0t = \{i \in S : i > t\}$.

Define $\omega_t = \alpha + \sum_{i<t}(e_i - x_i)$ and $\omega'_t = \alpha + \sum_{i<t}(e_i - y_i)$, with the convention that a summation over the empty set is zero. Also define $\alpha_t = \alpha + \sum_{i\le t}(e_i - x_i)$ and $\alpha'_t = \alpha + \sum_{i\le t}(e_i - y_i)$, so that $\alpha'_t - \alpha_t = (\omega'_t - \omega_t) + (\overline{x}_t - \overline{y}_t)$.

Step 1: Proving that $\omega'_t \le \omega_t$.

Suppose, by contradiction, that $\omega'_t > \omega_t$. This has three consequences. First,

$$P^0 t \backslash T \neq \emptyset, \tag{29}$$

Second, using the notation introduced at the beginning of the section,
$x_{P^0 t} = x^*(P^0 t;\ \alpha, \omega_t, \overline{x}_{T\cap P^0 t})$ and $y_{P^0 t} = x^*(P^0 t; \alpha, \omega'_t, \overline{x}_{T\cap P^0 t})$ (since $\overline{y}_{T\cap P^0 t} = \overline{x}_{T\cap P^0 t}$). It follows from Lemma 3 that

$$y_{P^0 t} < x_{P^0 t}, \tag{30}$$

where the inequality is strict because (9.12) guarantees that the consumption of at least one member of $P^0 t$ is not fixed. Third, $\alpha'_t > \alpha_t$. It follows from Lemma 2 that

$$y_{F^0 t} \ge x_{F^0 t} \tag{31}$$

whenever $F^0 t$ is nonempty since in that case $x_{F^0 t} = x^*(F^0 t; \alpha_t, \omega, \overline{x}_{T\cap F^0 t})$ and $y_{F^0 t} = x^*(F^0 t; \alpha'_t, \omega, \overline{x}_{T\cap F^0 t})$.

If either $F^0 t$ is empty or (9.14) is an equality, define the consumption plan y' for S by letting $y' = \overline{y}_t$ and $y'_i = x_i$ for every $i \neq t$. Note that y' satisfies all the constraints of the maximization problem whose solution is y. Yet, $y' > y$, a contradiction.

If $F^0 t$ is nonempty and (9.14) is not an equality, pick any $i < t$ such $y_i < x_i$ and the smallest $j > t$ such that $y_j > x_j$. Observe that

$$\sum_{k\le l}(y_k - e_k) < \sum_{k\le l}(x_k - e_k) \le \alpha \text{ whenever } i \le l < j. \tag{32}$$

Define y^ε by $y^\varepsilon_i = y_i + \varepsilon$, $y^\varepsilon_j = y_j - \varepsilon$, and $y^\varepsilon_k = y_k$ if $k \neq i, j$. In view of (9.15), we can choose $\varepsilon > 0$ small enough to ensure that y^ε satisfies all the constraints of the maximization problem whose solution is y. But by optimality of x and strict concavity of the benefit functions, we know that $b'_i(y_i) > b'_j(y_j)$. Therefore $\sum_k [b_k(y^\varepsilon_k) - b_k(y_k)] > 0$ for sufficiently small ε, contradicting optimality of y and completing Step 1.

Step 2. Proving that $\alpha'_t \geq \alpha_t$.

Suppose, again by way of contradiction, that $\alpha'_t < \alpha_t$. Note, first, that this inequality implies (9.12). Second, because of Lemma 2,

$$y_{F^0t} \leq x_{F^0t} \tag{33}$$

whenever F^0t is nonempty. Third, $\omega_t > \omega'_t$ and therefore, by Lemma 3,

$$y_{P^0t} > x_{P^0t}. \tag{34}$$

If either F^0t is empty or (9.16) is an equality, choose any $i < t$ such that $y_i > x_i$. We claim that

$$\sum_{k \leq l}(x_k - e_k) < \alpha - \theta_l \omega \text{ whenever } i \leq l. \tag{35}$$

To see why this is true, note that if $i \leq l < t$, (9.17) implies that $\sum_{k \leq l}(x_k - e_k) < \sum_{k \leq l}(y_k - e_k) \leq \alpha$. Next, if $t \leq l$, $\sum_{k \leq l}(x_k - e_k) = (\alpha - \alpha_t) + \sum_{t < k \leq l}(x_k - e_k) < (\alpha - \alpha'_t) + \sum_{t < k \leq l}(x_k - e_k) = \sum_{k \leq l}(y_k - e_k) \leq \alpha - \theta_l \omega$. Because of (9.18), the consumption plan x^ε defined by $x_i^\varepsilon = x_i + \varepsilon$ and $x_k^\varepsilon = x_k$ for $k \neq i$ satisfies the constraints of the problem whose solution is x when ε is small enough, a contradiction.

If F^0t is nonempty and (9.16) is not an equality, pick any $i < t$ such $y_i > x_i$ and the smallest $j > t$ such that $y_j < x_j$. Using virtually the same argument as to prove (9.18), we obtain

$$\sum_{k \leq l}(x_k - e_k) < \alpha \text{ whenever } i \leq l < j. \tag{36}$$

Define x^ε by $x_i^\varepsilon = x_i + \varepsilon$, $x_j^\varepsilon = x_j - \varepsilon$, and $x_k^\varepsilon = x_k$ if $k \neq i, j$. In view of (9.19), we can choose $\varepsilon > 0$ small enough to ensure that x^ε satisfies all the constraints of the maximization problem whose solution is x. But by optimality of y and strict concavity of the benefit functions, we know that $b'_i(x_i) > b'_j(x_j)$. Therefore $\sum_k [b_k(x_k^\varepsilon) - b_k(x$

Appendix 2

Proof of Lemma 5. Let S, T be two coalitions such that $S < T$. Let $\alpha := \sum_{i \in T}[x_i^*(S \cup T) - x_i^*(T)]$. Optimality requires $\alpha \geq 0$. Furthermore, $x_T^*(S \cup T) = x^*(T; \alpha, 0)$ while $x_S^*(S \cup T) = x^*(S; 0, \alpha)$. Invoking Lemmata 2 and 3 completes the proof. □

Proof of Lemma 6. Let $S, Q, T, \overline{x}_T$ and $\overline{y}_T$ satisfy the assumptions of the lemma. The case where $\overline{y}_T = \overline{x}_T$ being trivial, assume $\overline{y}_T < \overline{x}_T$. For every $i \in S \backslash T$,

$$d_i := x_i^*(S; \overline{y}_t) - x_i^*(S; \overline{x}_t) \geq 0 \tag{37}$$

because of Lemma 4, and

$$x_i^*(S; \overline{x}_T) - x_i^*(Q; \overline{x}_T) \geq 0, \tag{38}$$

as is easily seen by defining $\omega = \sum_{i \in S}(e_i - x_i^*(Q; \overline{x}_T))$, noting that $x^*(S; \overline{x}_T) = x^*(S; 0, 0, \overline{x}_T)$, $x_S^*(Q; \overline{x}_T) = x^*(S; 0, \omega, \overline{x}_T)$, and applying Lemma 3.

By strict concavity of the benefit functions, (9.20) and (9.21) imply that $b_i(x_i^*(S; \overline{y}_T)) - b_i(x_i^*(S; \overline{x}_T)) \leq b_i(x_i^*(Q; \overline{x}_T) + d_i) - b_i(x_i^*(Q; \overline{x}_T))$ for every $i \in S \backslash T$. Therefore,

$$\begin{aligned}
& v(S; \overline{y}_T) - v(S; \overline{x}_T) \\
= \; & \sum_{i \in S}[b_i(x_i^*(S; \overline{y}_T)) - b_i(x_i^*(S; \overline{x}_T))] \\
\leq \; & \sum_{i \in T}[b_i(\overline{y}_i) - b_i(\overline{x}_i)] + \sum_{i \in S \backslash T}[b_i(x_i^*(Q; \overline{x}_T) + d_i) - b_i(x_i^*(Q; \overline{x}_T))] \\
= \; & [\sum_{i \in T} b_i(\overline{y}_i) + \sum_{i \in S \backslash T} b_i(x_i^*(Q; \overline{x}_T) + d_i) + \sum_{i \in Q \backslash S} b_i(x_i^*(Q; \overline{x}_T))] \\
& -[\sum_{i \in T} b_i(\overline{x}_i) + \sum_{i \in S \backslash T} b_i(x_i^*(Q; \overline{x}_T)) + \sum_{i \in Q \backslash S} b_i(x_i^*(Q; \overline{x}_T))].
\end{aligned}$$

The second bracket is just $v(Q; \overline{x}_T)$. To complete the proof, we need only show that the first bracket does not exceed $v(Q; \overline{y}_T)$. We show that the consumption plan

$$x_i = \left\{ \begin{array}{cc} \overline{y}_i & \text{if } i \in T, \\ x_i^*(Q; \overline{x}_T) + d_i & \text{if } i \in S \backslash T, \\ x_i^*(Q; \overline{x}_T) & \text{if } i \in Q \backslash S \end{array} \right\} \tag{39}$$

satisfies the constraints of the maximization problem whose solution is $x^*(Q;\overline{y}_T)$. Since $x_T = \overline{y}_T$, all we have to check is that

$$\sum_{i \leq j}(x_i - e_i) \leq 0 \text{ for every } j \in Q. \tag{40}$$

If $j \in S$, combining (9.22) with (9.20) and (9.21) yields that $\sum_{i \leq j} x_i \leq \sum_{i \leq j} x_i^*(S;\overline{y}_T) \leq \sum_{i \leq j} e_i$. If $j \in Q \backslash S$, we get $\sum_{i \leq j} x_i = \sum_{i \in T} \overline{y}_i + \sum_{i \in S \backslash T} [x_i^*(S;\overline{y}_T) - x_i^*(S;\overline{x}_T)] + \sum_{i \in Pj \backslash T} x_i^*(Q;\overline{x}_T)$ $\leq \sum_{i \in T} \overline{x}_i + \sum_{i \in Pj \backslash T} x_i^*(Q;\overline{x}_T) = \sum_{i \leq j} x_i^*(Q;\overline{x}_T) \leq \sum_{i \leq j} e_i$. □

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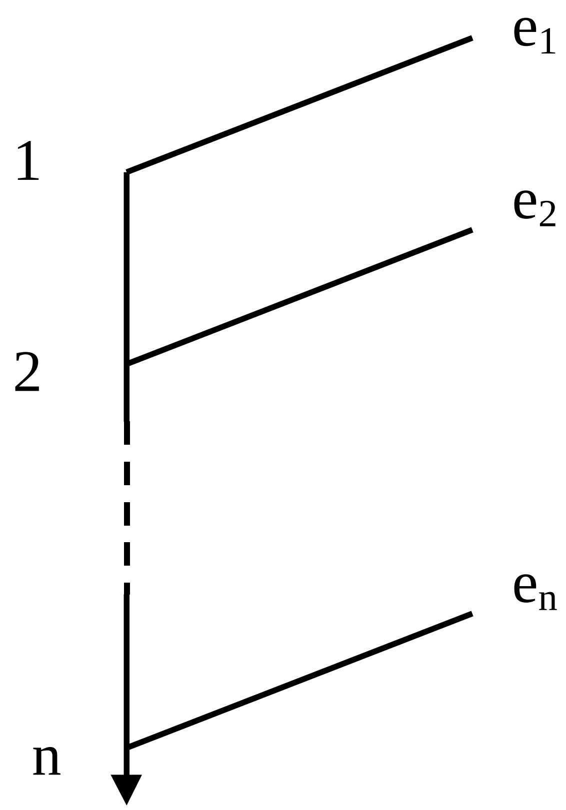

Figure 1