W87-19

NON-TRANSFERABLE UTILITY VALUES

OF VOTING GAMES*

Roy Gardner

January 1986
Revised, January 1987

*This research was supported by SFB 303 of the German Science Foundation, University of Bonn, and by the Alexander von Humboldt Foundation.

Paper to be presented at the World Conference on Game Theory, June 19, 1987.

ABSTRACT

A voting game is a non-transferable utility (NTU) game with a simple game structure. When the Shapley-Shubik index of a simple game is strictly positive, then the corresponding voting game has a strict NTU value. Moreover, the Shapley-Shubik index is the unique NTU value for a certain class of voting games. These results lead to a solution of the problem of a group choosing its leader.

1. Introduction.

The situation often arises in which a group must select one of its members for some position or office, the responsibilities and powers of which are indivisible. Such is the case for instance when an academic department elects its chairman or a legislative body elects its speaker. An extreme example occurred during the American Civil War, when some military units elected their commanders. As in any social choice problem, the outcome depends on the voting rule and the preferences of the individuals concerned. This paper analyzes these situations as cooperative games with non-transferable utility (NTU), called voting games, and proposes as their solution the NIU value.

Besides its social choice interest, the study of the NIU value of voting games has an additional motive. Recently a series of articles highly critical of the NIU value has appeared [4,6,7] . One response, as in Aumann [2], is to present instances where the NIU value leads to reasonable results. This paper shows that the NIU value of voting games gives results at least as reasonable as those of the Shapley-Shubik index [9] for simple games. Indeed, the Shapley-Shubik index is the NIU value of a certain class of voting games (Proposition 3). Moreover, a large class of voting games have only strict values (Proposition 1), even though these games are not covered by either of the axiomatizations of the strict NIU value which have recently appeared [1,5]. It appears that a comprehensive axiomatization of strict NIU values is still to come.

The paper is organized as follows. The next section formalizes voting games and their NIU values. Section 3 'studies general properties of the NIU values of such games. The final-two sections compute some specific examples where political power is evenly and unevenly distributed, respectively.

2. Model of the Officer Election Problem

Let N be a finite set of players, numbered 1,2,...,n. A coalition S is a subset of N. The group has to choose one of its members for a certain position, whose powers and responsibilities are indivisible. Each player has a von Neumann-Morgenstern utility over N. Let $p = (p_1,\ldots,p_n)$ be a lottery over the player set. Then player i's utility function $u_i(p)$ is assumed to satisfy:

$$u_i(p) = \sum_j u_{ij}p_j \tag{1}$$

$$u_{ii} = 1 \tag{2}$$

(2) is a normalization, implying that each player would like to be chosen for the position. Condition (1) requires that utility be additively separable across players. It is convenient to assume also that

$$\text{for all } i,j \quad u_{ij} = 0 \text{ or } 1. \tag{3}$$

(3) is restrictive; however, most of the results go though under the assumption that $u_{ij} \geq 0$. The motivation for (3) is that one can interpret outcomes in the model as the probability that a player approves of the group choice, $u_{ij} = 1$ denoting approval and $u_{ij} = 0$ denoting disapproval of j's choice by i. The matrix U with typical element u_{ij} therefore contains all relevant information about players' preferences.

The voting rules are assumed to induce a simple game structure with coalition function w. In particular, w is a strong, proper, monotonic simple game with at least the grand coalition winning. A player i is <u>pivotal</u> for coalition S when S is losing but $S \cup \{i\}$ is winning. The Shapley-Shubik index $\varphi w(\{i\})$ is the probability that player is pivotal for coalition S in a random ordering of N, when all random orderings are equally likely. Player i has a <u>veto</u> if he belongs to every winning coalition.

The voting game based on U and w has the coalition function v(S), telling what the members of S can assure themselves of. A losing coalition S can only assure its members of utility o. A winning coalition can choose the officeholder from among its own members. Players outside such a coalition have the fixed threat of not serving. No generality is lost by assuming that a winning coalition chooses a lottery p over its members. v(S) is thus given by:

$$v(S) = \begin{cases} \left\{ \begin{array}{l} u_i,\ i \text{ in } S:\ u \leq Up \\ u_i = p_i = o,\ i \text{ not in } S \end{array} \right\} & S \text{ winning} \\ \\ \{u_i = o, \text{ all } i\} & S \text{ losing.} \end{cases}$$

To compute an NTU value of the voting game v, introduce a vector of nonnegative weights $\lambda = (\lambda_1,\ldots,\lambda_n)$, at least one of which is positive. Define the transfer function $v_\lambda(S)$ such that

$$v\ (S) = \begin{cases} \max\limits_{u \in v(S)} \sum \lambda_i u_i & S \text{ winning} \\ \\ o & S \text{ losing.} \end{cases} \qquad (4)$$

Then an NTU value of the voting game is a utility vector u in v(N), such that

$$\varphi v_\lambda(\{i\}) = \lambda_i u_i \qquad (5)$$

for every player i, where φv_λ is the Shapley value of v_λ. Shapley's result [8] shows the existence of a solution to (5) for finite games.

3. Strict NTU Values of Voting Games

An NTU value is called strict if it is strictly positive. The strictness of the NTU value of a voting game is implied by the strict positivity of the Shapley-Shubik index of w. A player is called a dummy if his Shapley-Shubik index is o. One has the following:

Proposition 1. Suppose that no player in a voting game is a dummy. Then every NTU value of the voting game is strict.

Proof. It follows from (1)-(5) that $v_\lambda(N) > o$. Let S be the least coalition containing player i, such that $v_\lambda(S) > o$. Clearly, S must be winning. Now order the players in S in such a way that player i is pivotal. Then player i has a positive marginal product, hence a positive expected marginal product $\varphi v_\lambda(\{i\})$. (5) then implies that u_i and λ_i are positive. Since no player is a dummy, the argument holds for every player i and an NTU value must be strict.

It is interesting that one has here an entire class of games with strict values, despite their polyhedral character (which excludes them from the axiomatization in [1]). The result [5,Theorem 2.6] proving the existence of strict values for polyhedral games also does not apply to voting games, since they do not in general satisfy the condition of being normally closed. Take n = 2 and U = I, the identity matrix. Let player 1 be a dummy. Then the NTU values are of the form u' = (1,o), $\lambda = (\lambda_1, o)$, $\lambda_1 > o$. This also shows that Proposition 1 cannot be extended.

The next result gives bounds on any NTU value of a voting game:

Proposition 2. Suppose that no player is a dummy. Then any NTU value satisfies the following bounds:

$$\varphi w(\{i\}) \leq u_i \leq 1. \qquad (6)$$

Proof. The upper bound follows immediately from (1)-(3). As for the lower bound, player i pivots with probability $\varphi w(\{i\})$. Moreover, the value of his marginal product when he pivots is at least λ_i, since $v_\lambda(S) \geq \lambda_i$. Therefore the expected value of i's marginal product is at least

$$\lambda_i \varphi w(\{i\}) \leq \varphi v_\lambda(\{i\}) = \lambda_i u_i.$$

Since λ_i is positive, one has the lower bound in (6).

Of special interest is the case when the lower bound in (6) is attained for all players i; then the Shapley-Shubik index is itself an NTU value. The relevant result is the following:

Proposition 3. Suppose no player is a dummy, and U = I. Then the Shapley-Shubik index is the unique NTU value.

Proof. First, the Shapley-Shubik index is an NTU value. Let λ be a vector of 1's. Then $v_\lambda = w$, whence it follows that

$$\varphi v_\lambda = \varphi w = u.$$

If there were another NTU value, then for some player i, $u_i > \varphi w(\{i\})$. Since $u = Ip$ on the boundary of $v(N)$, one then faces the contradiction.

$$\sum u_i > 1 = \sum p_i = \sum u_i.$$

This result shows that the Shapley-Shubik is not restricted exclusively to transferable utility situations. Power can be indivisible, as in a voting game, and the index can still apply. What is crucial is that preferences be selfish, in the sense that each player wants to see himself (and only himself) in the position of power.

It is clear that even if there are dummy players, the Shapley-Shubik index is still an NTU value, although no longer strict. In such cases, there may be other NTU values.

4. NTU Values when Power is Equally Divided

This section computes explicit values for voting games when power is equally divided. Such games arise under unanimity rule, when only the grand coalition is winning, as well as all forms of majority rule, strict or qualified. Before proceeding, some additional notation is necessary. For each player i, let A_i denote the set of players that i approves of:

$$A_i = \{ j \in N: u_{ij} = 1 \}.$$

Let B_i be the set of players that approve of, or back, player i:

$$B_i = \{ j \in N: u_{ij} = 1 \} .$$

Finally, let P denote the set of strong Pareto optimal players. Player i is strongly Pareto optimal if there is no player j such that $B_j \supset B_i$ and the inclusion is strict.

No generality is lost by normalizing λ such that $v_\lambda(N) = 1$. One can now characterize an NTU value for any voting game based on unanimity rule:

<u>Proposition 4.</u> An NTU value for a voting game based on unanimity rule satisfies the following conditions:

$$\text{for every player i,} \quad \lambda_i \sum_{k \in A_i} p_k = 1/n \tag{7}$$

$$\text{for every i in P,} \quad \sum_{k \in B_i} \lambda_k = 1 \tag{8}$$

$$\sum_{k \in P} p_k = 1. \tag{9}$$

<u>Proof.</u> Consider the linear programming problem $\max_p \lambda U p$ subject to $p_i \geq 0$, $\sum p_i = 1$.

Denote the i-th column of U by U_i. The dual linear programming problem is min z, subject to $z \geq \lambda U_i$. If player i is Pareto inferior to player j, then $\lambda U_i < \lambda U_j$. In the primal, complementary slackness implies $p_i = o$. This establishes (9). The common value of both these programs is 1, from which (8) follows. (7) follows from the fact that

$$\varphi v_\lambda(\{i\}) = 1/n = \varphi w(\{i\}),$$

since $v_\lambda = w$.

Unanimity rule and majority rule are the same when there are two players; hence, so do their NTU values. Their NTU values continue to coincide when there are three players:

Proposition 5. For n = 3, the NTU value for the voting game based on unanimity coincides with that for the voting game based on majority rule. Moreover, these NTU values are unique.

Proof. It suffices to consider only those U for which no player i has backing $B_i = N$. Tables 1 and 2 summarize the relevant calculations. NTU values for other U follow from those given by symmetry. A routine calculation shows that these are the only NTU values.

It is possible to push Proposition 5 somewhat further. A sufficient condition for the coincidence of majority and unanimity rule voting game NTU values is that for every winning coalition S, $v_\lambda(S) = 1$. However, the coincidence cannot hold in general, as the following example shows. Let U be the symmetric matrix with $u_{12} = u_{22} = u_{45} = 1$, all other u_{ij} above the diagonal = o. One can show that the NTU value under unanimity rule is $u_1 = u_2 = u_3 = 18/3o$, $u_4 = u_5 = 12/3o$; whereas under majority rule, $u_1 = u_2 = u_3 = 17/3o$, $u_4 = u_5 = 13/3o$. The discrepancy arises because for a coalition like $\{1,2,4\}$, $v_\lambda(\{1,2,4\}) < 1$.

Call a voting game symmetric when some NTU value has u_i the same for all players. A class of symmetric voting games can be generated in the following manner. Arrange the players in order around a circle. Let each player approve of himself and the next k-1 players to his right. For instance, when n = 4 and k = 2,

$U = \begin{pmatrix} 1 & 1 & o & o \\ o & 1 & 1 & o \\ o & o & 1 & 1 \\ 1 & o & o & 1 \end{pmatrix}$ represents such preferences. It is easy to check that the NTU value for this voting game under either majority or unanimity rule is $u_i = 1/2$, with $\lambda_i = 1/2$ for all i.

This example can be generalized as follows. Suppose that for all players i, there exists a positive k such that

$$k = |A_i| = |B_i|, \qquad (|\cdot| \text{ denotes cardinality})$$

and moreover for each pair (i,j)

$$A_i \neq A_j \quad \text{and} \quad B_i \neq B_j.$$

Call such preferences <u>symmetric of order k</u>. One has the following:

<u>Proposition 6</u>. The voting game based on majority rule with symmetric preferences of order k is symmetric.

<u>Proof</u>. Let s be the size of the smallest majority. Consider first the case $s \geq k$. Take $\lambda_i = 1/k$. Then for any majority coalition, $v_\lambda(S) = 1$, since a majority must contain all the supporters of some Pareto player. $v_\lambda =$ w, and both are symmetric. One has the symmetric value $u_i = k/n$.

Next, consider the case $s < k$. Take the same λ_i as before. For S winning, one has $v_\lambda(S) = \min(|S|/k, 1)$, since all the members of S must support some Pareto player. v_λ is symmetric, and one has the same NTU value u_i as before.

It is clear that Proposition 6 holds for any form of qualified majority rule, including unanimity rule.

As a final example, call preferences <u>consecutive</u> if players can be ordered on a line in such a way that each player approves of the player to his immediate right, but no further. This nomenclature is inspired by that used in Greenberg and Weber [3]. One can imagine such an ordering taking place along an ideological spectrum for example. Under majority rule and an even number of players, consecutive preferences lead to a symmetric voting game. However, if the number of players is odd and greater than 3, consecutive preferences lead to an asymmetric voting game. For n = 5, the unique NTU value is $u_i = 1/3$, i odd and $u_i = 1/2$, i even. The intuition behind this is that the all odd-numbered coalition $\{1,3,5\}$ is composed of the two "extremists" incapable of supporting each other, or of supporting or being supported by player 3 in the middle. This cannot happen to any majority containing an even-number player; at least one player must have another supporter.

5. NTU Values with a Veto Player

This section studies voting games where one player has veto power, with the remaining power being distributed evenly among the other players. In particular, suppose player 1 has this veto, and that any coalition containing player 1 and having at least q members, $1 < q < n$, is winning. The Shapley-Shubik index is:

$$\Psi w(\{1\}) = (n-q+1)/n$$

$$\Psi w(\{i\}) = (q-1)/n(n-1), \quad \text{for } i > 1.$$

Player 1 is considerably more powerful than the rest.

It is instructive to consider first the case $q = 2$, $n = 3$, and compare the results obtained with those of Proposition 5. One has the following:

<u>Proposition 7</u>. For $q = 2$, $n = 3$, the veto player does better than his counterpart without veto power, or in both cases his NTU value $u_1 = 1$.

<u>Proof</u>. The relevant calculations are summarized in Table 3, whose results should be compared to those of Tables 1 and 2.

It is interesting to compare the NTU values for $U = I$ and $U = \begin{pmatrix} 1 & 1 & o \\ o & 1 & o \\ o & o & 1 \end{pmatrix}$ in Table 3. These situations differ solely in the value of u_{12}. One can interpret this difference as player 1's <u>throwing his support</u> to player 2. In terms of either the probability of being elected or the probability of approving of the outcome, this support from player 1 is worth $5/6-1/6 = 2/3$ to player 2. Indeed, this notion of the value of additional support makes sense in any context where a single u_{ij} changes from o to 1, or vice versa. Even though utility is non-transferable, the value of such support can be measured.

To see the effect of the power asymmetry on an otherwise symmetric situation, suppose preferences are symmetric of order 2, q = 3, and n = 4:

$$U = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The NTU value is u' = (3/4, 1/2, 1/4, 1/2), with λ = (2/3,1/3,2/3,1/3). The veto player is now most likely to approve of the outcome. Even among the weaker players, the NTU value is no longer symmetric. Player 2 does better because of player 1's backing, while player 4 does better because of his support for player 1.

As a final example, suppose the power distribution is the same as that just given, but preferences are consecutive. The NTU value is u'= (2/3,2/3,1/3,1/3), with λ = (3/4,1/4,1/2,1/2). Again, player 2 is benefiting from the veto player's support.

It pays to have veto power and to back or be backed by someone with veto power.

ACKNOWLEDGEMENTS

The author wishes to thank R. Aumann, R. Becker, P. Coughlin, B. Peleg, and R. Selten for their helpful comments. The present proof of Proposition 1 was suggested by B. Peleg. R. Aumann proposed the example leading to symmetric preferences of order k.

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Table 1. NTU values for majority rule voting games, n = 3.

U	λ	p	u
1 0 0		1/3	1/3
0 1 0	(1,1,1)	1/3	1/3
0 0 1		1/3	1/3
1 1 0	(1/2,1/2,1)	0	2/3
0 1 0		2/3	2/3
0 0 1		1/3	1/3
1 1 1		0	1
0 1 0	(1/3,2/3,2/3)	1/2	1/2
0 0 1		1/2	1/2
1 1 0		0	1/2
0 1 1	(1/2,1/2,1/2)	1/2	1
0 0 1		1/2	1/2
1 1 0		a*	2/3
1 1 0	(1/2,1/2,1)	2/3−a	2/3
0 0 1		1/3	1/3
1 1 1		b**	1
0 1 0	(1/3,2/3,2/3)	1/2	1/2
1 0 1		1/2−b	1/2
1 0 1		1/3	2/3
1 1 0	(1/2,1/2,1/2)	1/3	2/3
0 1 1		1/3	2/3

$*0 \leq a \leq 2/3$.

$**0 \leq b \leq 1/2$.

Table 2. NTU values for unanimity rule voting games, n = 3.

U	λ	p	u
1 0 0		1/3	1/3
0 1 0	(1,1,1)	1/3	1/3
0 0 1		1/3	1/3
1 1 0		0	2/3
0 1 0	(1/2,1/2,1)	2/3	2/3
0 0 1		1/3	1/3
1 1 1		0	1
0 1 0	(1/3,2/3,2/3)	1/2	1/2
0 0 1		1/2	1/2
1 1 0		0	1/2
0 1 1	(2/3,1/3,2/3)	1/2	1
0 0 1		1/2	1/2
1 1 0		a*	2/3
1 1 0	(1/2,1/2,1)	2/3−a	2/3
0 0 1		1/3	1/3
1 1 1		b**	1
0 1 0	(1/3,2/3,2/3)	1/2	1/2
1 0 1		1/2−b	1/2
1 0 1		1/3	2/3
1 1 0	(1/2,1/2,1/2)	1/3	2/3
0 1 1		1/3	2/3

$*0 \le a \le 2/3.$

$**0 \le b \le 1/2.$

Table 2. NTU values for the veto player games, n = 3. (Player 1 has a veto.)

π	λ	p	u
1 0 0 0 1 0 0 0 1	(1,1,1)	2/3 1/6 1/6	2/3 1/6 1/6
1 1 0 0 1 0 0 0 1	(4/5,1/5,1)	0 5/6 1/6	5/6 5/6 1/6
1 0 0 0 1 1 0 0 1	(1,1/2,1/2)	2/3 0 1/3	2/3 1/3 1/3
1 1 1 0 1 0 0 0 1	(2/3,1/3,1/3)	0 1/2 1/2	1 1/2 1/2
1 0 0 1 1 1 0 0 1	(5/6,1/6,5/6)	4/5 0 1/5	4/5 1 1/5
1 1 0 0 1 1 0 0 1	(4/5,1/5,1)	0 5/6 1/6	5/6 5/6 1/6
1 1 0 1 1 0 0 0 1	(4/5,1/5,1)	a* 5/6−a 1/6	5/6 5/6 1/6
1 1 1 0 1 0 1 0 1	(2/3,1/3,1/3)	b** 1/2 1/2−b	1 1/2 1/2
1 0 1 0 1 0 1 1 1	(5/6,5/6,1/6)	c*** 1/5 4/5−c	4/5 1/5 1
1 0 1 1 1 0 0 1 1	(4/5,1/5,1/5)	2/3 1/6 1/6	5/6 5/6 5/6

*$0 < a < 5/6$.

**$0 < b < 1/2$.

***$0 < c < 4/5$.