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COMMON PROPERTY RESOURCES: CAN EVERYONE BENEFIT FROM GROWTH?

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1. Introduction

We consider surplus-sharing problems arising when some common property resources must be jointly utilized for the benefit of all. These resources can be used directly for consumption, as in the familiar Cake Division problem, or used as inputs in a production process (think of the extraction of publicly owned exhaustible resources); or they could be the technology of production itself as when some commodities are supplied by a regulated monopoly, or when every coalition of agents has access to the technology for producing a public good - see Foley [1967] -).

The point of common property is that the resources should be used to the common benefit. This does not prescribe any precise distribution rule, yet it does constrain our allocation methods. One compelling consequence (perhaps the only one) of common property is the Resource Monotonicity (RM) property, requiring that growth of the jointly owned resources should benefit to everyone, or at least that the utility of no agent should decrease when the common property resources increase. If we Interpret common property as giving to each agent the power to veto any utilization of the resources, then surely an allocation of growing resources violating the RM axiom would face a veto. Even a less stringent interpretation of common property is contradicted when an agent suffers a utility loss as the value of his property increases objectively. For a more detailed discussion of Resource Monotonicity and public ownership see Roemer [1986], Moulin and Roemer [1986], Roemer and Silvestre [1987],

Resource Monotonicity alone is always compatible with first best Pareto optimality. Indeed pick some fairly arbitrary cardinal utility representation of individual preferences, and consider the <u>welfare egalitarian</u> outcome(s) achieving the highest feasible egalitarian utility vector (that is, all agents end up with the same cardinal utility level). This, barring a technical difficulty known as the dilemma equality-efficiency, defines a Pareto optimal solution. When resources increase, the feasible utility set expands and the Pareto optimal egalitarian utility vector increases, too, so this solution is resource monotonic.

In most surplus-sharing problems, however, we must incorporate an Individual Rationality (IR) constraint, setting a lower bound on Individual utilities. This secure utility level is interpreted as a disagreement pay-off in Nash's axiomatic bargaining approach. In the more general perspective of cooperative games, it measures the opportunity cost of cooperation to each individual agent. In the models discussed below, the secure utility level results from individual rights over the common property resources: for instance each agent is entitled to the 1/n-th share of the consumption goods (case of the Cake Division problem) or each agent can freely use the technology for himself (case of the production economy).

Within the Individual Rationality bounds, Resource Monotonicity may be hard to meet, as growth also raises every agent's secure utility level. The existence of a Pareto optimal surplus-sharing rule satisfying both Resource Monotonicity and Individual Rationality is the principal question addressed in this paper.

Consider the problem of dividing a vector of commodities (cake) among a given set of agents with equal claims on it. Production and borrowing are ruled out. This much studied model (see e.g. Thomson and Varian [1986]) is the general form of the Cake Division problem. The equal claim assumption naturally suggests the secure utility level whereby an agent gets the n-th share (n is the number of agents) of each commodity. Resource Monotonicity, on the other hand, follows from joint ownership: if agents get to eat a bigger cake, this objective improvement should not lower anyone's utility. That RM and IR may not be compatible with a Pareto optimal Division of the cake was shown in Moulin and Thomson [1987]. Our general results will be systematically applied to Cake Division, thereby throwing more light on this difficulty: see example 5 in Section 4, Lemma 1 and Example 6 in Section 5, Examples 7, 9 and Lemma 4 in Section 6.

A second focal example for the RM axiom is the production economy where the technology is publicly owned. In the archetypal story discussed in Moulin and Roemer [1986] the n agents own their labor and can produce corn from labor by means of an increasing returns to scale technology. The RM axiom is then formulated as Technological Monotonicity: if the technology improves upon (i.e. the production set expands) the utility of no agent decreases. The IR constraint is dictated by the public character of the technology: each agent has free access to it, so by running the production process at his own cost he can secure a certain utility level. Since returns to scale are increasing the vector of secure utility levels can be improved upon by cooperation, whence the surplus-sharing issue.

A third example is the remuneration of the inputs of a (publicly owned) production process. Each agent brings his own "skill" to the joint production; he derives no utility from consuming his skill. The most radical supporters of egalitarianism (e.g. Dworkin [1981] Cohen [1984]) propose that individual talents should not be viewed as private property but rather as a resource to be utilized for the sake of common welfare. With nontransferable inputs, the simplest expression of the common property of those inputs is, again, a monotonicity property: when the skill of any single agent increases, the utility of none decreases. Individual Rationality, on the other hand, expresses the right of an agent to use privately his own skill in some independent production process: the secure utility level measures the opportunity cost of cooperation. In this model the tension between RM and IR reflects the dual interpretation of the skill inputs, as public assets when they enter the joint production process, as private endowments when one compute individual opportunity costs. We explore here a very simple version of this model (Example 8 in Section 6) leaving a more systematical study for future research.

Our goal in this paper is methodological. We propose an abstract model where the compatibility of the RM axiom, the IR axiom, and Pareto Optimality can be systematically explored. This model encompasses all three examples above, and more. It consists of a surplus-sharing problem (an ordinal version of axiomatic bargaining) where both the feasible set and the vector of secure utility levels (disagreement point) depend upon a common parameter λ (interpreted as the common property resource). The set of parameters is ordered in such a way that when λ increases, the set of feasible (cooperative) utility vectors expands, and also the secure utility level of each agent rises. Thus when λ increases cooperative opportunities are better but at the same time the individual rationality constraint is tighter. Does it exist, then, a solution of each surplus-sharing problem (Pareto optimal and individually rational) that is increasing with respect to the parameter λ?

From the mathematical standpoint, this question is far from simple. We propose some necessary, as well as some sufficient conditions for existence. The latter require, however, that the variation of the feasible set w.r.t. the resource parameter satisfies additional lattice theoretical assumptions. Applying those results to the three examples described above unifies several earlier results on Resource Monotonicity and provides many new insights. Thus, at the cost of a higher level of abstraction we uncover deep structural similarities between micro-economic models as different as - say - the Cake Division problem and collusion in a quantity setting oligopoly (see below).

The general model is defined in Section 2 as an axiomatic bargaining problem parametrized by a "resource" λ varying in a lattice. Then we investigate the existence of a monotonic solution in two steps.

In Sections 3 and 4 we restrict attention to the welfare egalitarian solutions, that pick in the feasible set the highest utility vector along a given, fixed, monotone utility path. Up to rescaling accurately individual utilities, those solutions simply choose an efficient, equal utility vector. A welfare egalitarian solution being automatically resource-monotonic (see above), the problem is to contruct the monotone utility path in such a way that the highest feasible point on the path always lies above the secure utility vector. In Section 3 we describe two lattice properties of the feasible set correspondence, guaranteeing that the only resource monotonic solutions are welfare egalitarian. Those properties are the common key to several earlier results in production economies (Moulin [1987], Moulin and Roemer [1986], Moulin [1986]) , they are also applied here to a model of extraction of common property resources and to collusion in a quantity setting oligopoly (the latter in Section 7). In Section 4 we characterize the existence of such a welfare egalitarian solution in the two person case (Theorem 3).

In Sections 5, 6, 7 we address the general existence problem. A sufficient existence condition is that a take-all solution (where one agent keeps all the surplus above the secure utility vector) be monotonic. We use the Cake Division problem as an illustration (Example 6 and Lemma 1).

In Section 6, we derive a simple *necessary* condition for the existence of a monotonic solution in a *two* person problem. When the feasible set correspondence satisfies a certain lattice property called supermodularity, this condition turns out to be sufficient also. We apply this result to the remuneration of inputs problem and to Cake Division, thereby strengthening the previous results in Moulin and Thomson [1987] (see Lemma 4).

In Section 7 we study deficit-sharing (instead of suprlus-sharing) by similar methods. Here the bound on individual utility levels is an upper bound (instead of a lower bound when we speak of a secure utility level). An example is the production economies where the agents share a technology with decreasing returns to scale: as joint production creates a negative externality one does not want any agent to end up with a higher utility than when he runs alone the production function. Another example is collusion in an oligopoly where we view the demand function as the common property that the cartel must jointly exploit. A natural upper bound on the profit of any firm is its monopoly profit (that it would achieve in the absence of any competitor): see Example 11.

Resource Monotonicity is equally appealing in deficit-sharing as it is in surplus-sharing, and it raises a very similar mathematical question. In Section 7 we explain how the successive results for surplus-sharing can be adapted to deficit-sharing as well. Concluding comments are gathered in Section 8. A mathematical appendix contains all the proofs as well as a general characterization result for the two-person existence problem.

2. The Model

Denote by n the number of agents. The individual utility levels vary within a fixed interval [0,L] where L is positive and possibly too. The equality of individual upper bounds is not restrictive since our model is completely ordinal (see Remark 1 below).

A surplus-sharing situation (also called a bargaining situation in the literature) is a pair (d,S) where S is the feasible utility set and d is the secure utility vector (disagreement point). We assume:

. S is a closed, comprehensive, non-empty subset of $[0,L]^n$

. d is an element of S

If $L < +\infty$, the topology on $[0,L]^n$ is the usual topology of an euclidian space. If $L = +\infty$ we choose the extension of the euclidian topology that makes $[0,L]$ a compact set. Thus the set S is always compact.

Given a feasible utility set S, we denote by ∂S its Pareto set and by $\partial_w S$ its weak Pareto set defined as follows:

$$[\text{for all } u \in S,\ u \notin \partial_w(S) \ \text{ iff}\{\ \exists\ v \in S,\ v \neq u \text{ s.t. for all } i = 1,\ldots,n$$
$$v_i > u_i \text{ or } v_i = u_i = L\} \qquad (1)$$

or equivalently

$$\text{for all } u \in S,\ u \in \partial_w(S) \quad \text{iff}\{\forall v \in S,\ v \neq u,\ \ \exists\ i=1,\ldots,n$$
$$\{v_i < u_i < L \text{ and/or } v_i < u_i\}\} \qquad (2)$$

Note that if $L = +\infty$ this is the usual definition of the weak Pareto set. If L is bounded, the above definition is slightly more restrictive than the usual one.

We denote by $\sum(n)$ the set of surplus-sharing situations.

The resource paramter λ varies in a lattice $(\Lambda, \geq)$. A lattice is a partially ordered set in which any two elements have an infimum (greatest lower bound) and a supremum (smallest upper bound). We interpret the inequality $\lambda < \mu$ as: the resources increase from λ to μ.

That Λ is often only partially ordered should be clear from the examples outlined in Section 1: for instance in the Cake Division problem, Λ is the vector of commodities to be divided, and the ordering is the partial ordering of R_+^p. Actually we will show (see below Example 1) that if the ordering of Λ is complete, then the existence of a monotonic surplus-sharing solution holds trivially.

Definition 1.

A monotonic surplus-sharing problem is a mapping Φ from Λ into $\sum(n)$ associating to every resource parameter λ a surplus-sharing situation $\Phi(\lambda) = (d(\lambda), S(\lambda))$ and such that:

$$\text{for all } \lambda, \mu\varepsilon\Lambda : \{\lambda < \mu\} \Rightarrow \{d(\lambda) < d(\mu) \text{ and } S(\lambda) \subseteq S(\mu)\} \tag{3}$$

Thus the growth from λ to μ raises the secure utility vector and expands the feasible utility set. Our problem is to find a (Pareto optimal) surplus-sharing method that responds monotonically to an increase of the resources.

Definition 2.

Given a monotonic suprlus-sharing problem Φ, a monotonic solution is a mapping f from Λ into $[0,L]^n$ associating with each resource parameter λ a utility vector $f(\lambda)$ such that:

$$\text{for all } \lambda : f(\lambda) \varepsilon \partial S(\lambda) \text{ and } d(\lambda) < f(\lambda) \tag{4}$$

$$\text{for all } \lambda, \mu : \{\lambda < \mu\} \Rightarrow \{f(\lambda) < f(\mu)\} \tag{5}$$

A monotonic weak solution f is defined similarly with $f\varepsilon\partial_w S$ instead of $f\varepsilon\partial S$. We denote by $M(\Phi)$ (resp. $M_w(\Phi)$) the set of monotonic solutions (resp. monotonic weak solutions) of Φ.

Property (4) says that for each surplus-sharing situation, f picks a Pareto optimal utility vector and respects the Individual Rationality constraint.

Property (5) is Resource Monotonicity. We wish to study the sets $M(\Phi)$ and $M_w(\Phi)$.

Remark 1. For the sake of exposition, we choose to state our main problem (namely the existence of a monotonic solution) with (cardinal) utility vectors, like in the axiomatic bargaining approach. However, the whole model is purely ordinal. If we rescale agent 1's individual utility from u_1 to $\theta_1(u_2)$ - while keeping every other utility fixed, the set $M(\Phi)$ is transformed similarly: a mapping $f = (f_1,\ldots,f_n)$ is a monotonic solution before rescaling if and only if the mapping $(\theta_1(f_1),f_2,\ldots,f_n)$ is a monotonic solution after rescaling. In particular we can (and will) always choose the cardinal utility functions that have their values in a given interval $[0,L]$.

The ordinal character of our model is essential for its application to production economies and the Cake Division problem.

3. Welfare Egalitarian Solutions

In this section and the following, we study a class of solutions to the surplus-sharing problems that have been discussed extensively in the bargaining literature (see Thomson and Myerson [1980]). They are constructed with the help of a monotone utility path.

Definition 3

A monotone utility path is a mapping γ from $[0,L]$ into $[0,L]^n$ and such that:

i) $\gamma(0) = (0,\ldots,0)$; $\gamma(L) = (L,\ldots,L)$

ii) for all t, s : $\{t < s\} \Rightarrow \{\gamma(t) < \gamma(s)\}$ (6)

iii) for all $t \in [0,L]$: $\sum_{i=1}^{n} \gamma_i(t) = n.t$

A montone utility path may not have all its components $\gamma_i(t)$ strictly increasing for all t (see Figure 1); yet property (6) iii guarantees that for all t at least one such component strictly increases.

Fix a monotone utility path γ and consider a monotonic surplus-sharing problem Φ (Definition 1). Assume first that for each feasible set $S(\lambda)$, the sets $\partial S(\lambda)$ and $\partial_w S(\lambda)$ coincide (this assumption is not necessary for Theorem 1 below). The path γ, then, pierces the Pareto set $\partial S(\lambda)$ in exactly one point, denoted $\gamma \cap \partial S(\lambda)$ - see Figure 1.a - . By construction the mapping $f(\lambda) = \gamma \cap \partial S(\gamma)$ is monotonic w.r.t. λ : if λ increases, the set $S(\lambda)$ exapnds, therefore the intersection of $\partial S(\lambda)$ with γ occurs at the same point or at a higher point on γ : see Figure 1.b. Notice that if γ is strictly increasing in every component (which we do not necessarily assume) then one can rescale individual utilities so as to transform γ into the diagonal of $[0,L]^n$, whereby the intersection $\gamma \cap \partial S(\lambda)$ simply is the highest feasible equal utilities vector. Whence the name of welfare-egalitarian solutions.

When we drop the simplifying assumption that weakly Pareto optimal utility vectors are Pareto optimal as well, then the intersection of γ with the weak Pareto frontier $\partial_w S(\lambda)$ is non empty. We denote by $\gamma \cap \partial_w S(\lambda)$ the highest point (in the range of γ) of this intersection (see Figure 1.c). Thus defined, the mapping $\gamma \cap \partial_w S(\lambda)$ is still monotonic w.r.t. λ (see Lemma 7, 8 in the Appendix).

Of course, for an arbitrary choice of γ, the corresponding intersection $\gamma \cap \partial S(\lambda)$ defines a (monotonic) solution of the surplus-sharing problem Φ only if it safisfies Individual Rationality.

Definition 4

Given a monotonic surplus-sharing problem Φ, a welfare egalitarian solution of Φ (resp. a welfare egalitarian weak solution) is a monotonic solution f (Definition 2) such that there exists a monotone utility path γ

(Definition 3) satisfying

$$\text{for all } \lambda\varepsilon\Lambda : f(\lambda) = \gamma \cap \partial S(\lambda) \quad (7)$$

$$(\text{resp. for all } \lambda\varepsilon\Lambda : f(\lambda) = \gamma \cap \partial_w S(\lambda))$$

We denote by WE(Φ) (resp. $WE_w(\Phi)$) the set of welfare egalitarian solutions of Φ (resp. weak solutions).

Notation: $\lambda \vee \mu$ is the supremum of λ and μ in the lattice Λ, and $\lambda \wedge \mu$ is their infimum.

Theorem 1

Given is a monotonic surplus-sharing problem Φ, satisfying the following property:

$$\text{for all } \lambda, \mu\varepsilon\Lambda : S(\lambda \vee \mu) = S(\lambda) \cup S(\mu) \quad (8)$$

Then WE(Φ) = M(Φ), in other words any monotonic solution of Φ (if any) must be welfare egalitarian.

Theorem 2

Given is a monotonic surplus-sharing problem Φ satisfying the following property

$$\text{for all } \lambda, \mu\varepsilon\Lambda : S(\lambda \wedge \mu) = S(\lambda) \cap S(\mu) \quad (9)$$

Suppose moreover that

$$\text{for all } \lambda\varepsilon\Lambda \quad \partial S(\lambda) = \partial_w S(\lambda) \quad (10)$$

Then WE(Φ) = M(Φ), so any monotonic solution of Φ (if any) must be welfare egalitarian.

Any one of the lattice-homomorphism properties (8) (9) implies that only welfare egalitarian solutions achieve a monotonic surplus-sharing.

The complete proof of Theorems 1, 2 is given in the Appendix. We give here the intuition of these results. Assume (8) and pick a monotonic solution f of Φ : $f \in M(\Phi)$. We need to prove that f is actually welfare egalitarian. Pick any two λ, μ in Λ. Since f is monotonic, we have:

$$f(\lambda),\ f(\mu) \leq f(\lambda \vee \mu)$$

By (8), $f(\lambda \vee \mu)$ belongs to $S(\lambda)$ and/or to $S(\mu)$ - say it belongs to $S(\lambda)$. Since $f(\lambda)$ is Pareto optimal in $S(\lambda)$ we deduce $f(\lambda) = f(\lambda \vee \mu)$, whence $f(\mu) \leq f(\lambda)$. Thus for any two λ, μ we have $f(\mu) \leq f(\lambda)$ and/or $f(\lambda) \leq f(\mu)$. As λ varies, the values $f(\lambda)$ are therefore arrayed along a monotone path (see Lemma 9 in the Appendix). The proof of Theorem 2 is similar.

Note that the Individual Rationality constraint (the function $d(\lambda)$) plays no role in Theorems 1, 2. Therefore we can not tell in general whether a monotonic (welfare egalitarian) solution of Φ exists at all. However, in problems involving only <u>two</u> agents we will show that property (9), together with the monotonicity of $d(\cdot)$, does imply the nonemptiness of $WE(\Phi)$: see Corollary to Theorem 3 in the next section.

We now apply Theorems 1 and 2 to several examples.

<u>Example 1</u>: <u>Where the ordering of Λ is complete</u>.

Suppose any two parameters λ, $\mu \in \Lambda$ can be compared. If $\lambda > \mu$ then $\lambda \vee \mu = \lambda$ and $\lambda \wedge \mu = \mu$. Properties (8) (9) then follow from the monotonicity of S (assumption (3)). Hence Theorem 1 applies. Moreover the set $WE(\Phi)$ of welfare egalitarian solutions is non empty.

Indeed denote by Δ the range of the function d in $[0,L]^n$

$$u \in \Delta \Leftrightarrow \exists\ \lambda \in \Lambda : u = d(\lambda)$$

Since λ is completely ordered, Δ is a <u>chain</u> for the (partial) ordering of

$[0,L]^n$ (for any two u, $v \in \Delta$ we have $u > v$ and/or $v > u$), therefore it can be extended into a monotone utility path γ (see Lemma 9 in the Appendix). Now consider the intersection $f(\lambda)$ of γ with the Pareto frontier of $S(\lambda)$ (if $\partial S(\lambda) = \partial_w S(\lambda)$ then $f(\lambda)$ is the unique intersection of γ with $\partial S(\lambda)$; if $\partial S(\lambda) \neq \partial_w S(\lambda)$, then $f(\lambda)$ is the highest point in $\gamma \cap \partial_w S(\lambda)$; see Lemma 7). By construction of γ, we have $f(\lambda) > d(\lambda)$; moreover $f(\cdot)$ is monotonic w.r.t. λ (Lemma 8) so that f is indeed a welfare egalitarian solution of Φ (if $\partial S(\lambda) = \partial_w S(\lambda)$; otherwise it is a welfare egalitarian weak solution of Φ).

Thus when the set of resource parameters is completely ordered (e.g. the quantity of a single good), we can always share monotonically the surplus of a monotonic problem. All the subsequent examples involve a partially ordered set of resources.

Example 2: Axiomatic bargaining

Here Λ is the set of convex, compact and comprehensive subsets of R_+^n. Its lattice ordering is the set theoretic inclusion. The infimum $\lambda \wedge \mu$ is simply the intersection of λ and μ, whereas the supremum $\lambda \vee \mu$ is the convex hull of their union. Defining $S(\lambda) = \lambda$ and $d(\lambda) = 0$ we are in the framework of axiomatic bargaining. A monotonic solution of Φ selects a Pareto optimal element in every subset $\lambda \in \Lambda$, in such a way that the utility of no agent decreases when the feasible set λ expands. The basic result in Thomson and Myerson [1980] amounts to say that a monotonic solution must be welfare egalitarian as well. This result is also a Corollary of our Theorem 2 (since $S(\lambda) = \lambda$ satisfies (9)) with some minor qualifications (Thomson and Myerson use a stronger monotonicity assumption while we need to impose assumtion (10)).

Example 3. Extraction of common property resources.

Our n agents are fishing from common grounds. The effort spent by agent i in fishing is measured by a single parameter x_i. The total catch is $\lambda(\bar{x})$ where $\bar{x} = \sum_{j=1}^{n} x_j$ is the total fishing effort. Each agent receives a share proportional to his own effort: agent i's share is $y_i = (x_i/\bar{x}) \cdot \lambda(\bar{x})$. Finally agent i's utility is $u_i(y_i, x_i)$ where u_i is increasing in y_i and decreasing in x_i.

The resource is the catch function λ, and the partial ordering of two such functions is $\{\lambda > \mu\}$ iff $\{\lambda(x) > \mu(x)$ for all $x\}$. We think of our agents trying to agree upon some joint fishing policy achieving a fair and Pareto optimal outcome (see e.g. Dasgupta and Heal [1979]). The interpretation of the Resource Monotonicity axiom is straightforward.

Several definitions of the secure utility levels of individual agents are possible (see below). The assumptions of Theorems 1 and 2, however, do not depend on the definition of Individual Rationality.

We check that (8) holds true. Given a production function λ, the corresponding feasible utility set is defined as follows:

$$\{u \varepsilon S(\lambda)\} \text{ iff } \{ \exists (x_1,\dots,x_n) \quad u_i = u_i((x_i/\bar{x})\cdot\lambda(\bar{x}), x_i) \text{ for all } i\}$$

If λ, μ are two production functions, any outcome feasible with the production function ν, $\nu(x) = \sup\{\lambda(x), \mu(x)\}$ must be feasible with λ, or with μ or with both. Thus

$$S(\lambda \vee \mu) \subseteq S(\lambda) \cup S(\mu)$$

The converse inclusion is clear, whence (8). Thus Theorem 1 applies.

The weakest Individual Rationality requirement simply sets the secure utility level where no fishing is at all possible:

$$\bar{d}_i(\lambda) = u_i(0,0)$$

Then many welfare egalitarian solutions exist: to construct one, simply choose a cardinal utility representation $\bar{u}_i$ of preferences such that $\bar{u}_i(0,0) = 0$ (for instance $\bar{u}_i(y_i, x_i) = u_i(y_i, x_i) - u_i(0,0)$), then pick an equal utility Pareto optimal outcome (existence here is guaranteed if preferences are continuous and strictly decreasing in effort).

A second, more demanding, Individual Rationality requirement can be imposed when production exhibits increasing returns to scale: $\lambda(x)/x$ is nondecreasing in x. This assumption is reasonable when the total catch is a small fraction of the existing stock, and fishermen share information about successful catch. In general, think of a Research and Development process where the other agents' R.D. effort is a positive externality. For the rest of the Example, the set Λ consists of all production functions λ with nondecreasing returns to scale. Notice that Λ is stable by the supremum operation, so that (8) holds true on this domain, too.

With increasing returns to scale, each agent can be given free access to the production technology without exhausting the surplus opportunities. Define the secure utility levels as follows:

$$d_i(\lambda) = \max_{x_i \geq 0} u_i(\lambda(x_i), x_i) \qquad (11)$$

Then the utility vector $d(\lambda) = (d_1(\lambda), \ldots, d_n(\lambda))$ is always feasible and can generally be improved upon by cooperation : $d(\lambda) \in S(\lambda)$.

It turns out that this choice of the Individual Rationality constraint leaves exactly one welfare egalitarian solution of our joint production problem. Indeed consider the production functions $\lambda_a(x) = a \cdot x$ with constant returns to scale. The corresponding secure utility vector is then Pareto optimal as well (under CRS, utilization of the technology by other agents does not affect me):

$$d(\lambda_a) \in \partial S(\lambda_a) \tag{12}$$

Now consider a welfare egalitarian solution f of $\Phi = (S,d)$ with associated monotone utility path γ. Property (12) implies that the utility vector $d(\lambda_a)$ belongs to the range of γ. But as a increases from 0 to $+\infty$, $d(\lambda_a)$ defines a monotone utility path as well that coincides with γ (see the details in Moulin [1987]). Hence $d(\lambda_a)$ defines the unique welfare egalitarian solution f^* of our problem, that we call the Constant Returns Equivalent solution:

$$\text{for every } \lambda \text{(exhibiting IRS), } f^*(\lambda) = d(\lambda_{a^*}) \text{ where } a^* \text{ is the largest number such that } d(\lambda_a) \in S(\lambda) \tag{12}$$

Of course one must prove that the utility vector $f^*(\lambda)$ thus defined satisfies Individual Rationality ($f^*(\lambda) \geq d(\lambda)$). This is done by the same argument as in Lemma 2 of Moulin [1987a].

To summarize, we have shown that f^* is the only welfare egalitarian solution of Φ, and Theorem 1 shows that all monotonic solutions of Φ are welfare egalitarian. Hence f^* is the only monotonic solution of Φ : $M(\Phi) = \{f^*\}$.

This result is a variant of the main theorm in Moulin [1987a], explained in the next example. In Example 10 (Section 7), we adapt it to the case of a production function with <u>decreasing</u> returns to scale: there the utility level resulting from free access to the technology is viewed as an upper bound to an agent's final utility.

<u>Example 4.</u> <u>Production economy with one input and one output (Moulin [1987a]).</u>

Our n agents share a production function λ transforming labor (x) into corn (y). If agent i contributes x_i to the input and receives a share y_i of output, his final utility is $u_i(y_i, x_i)$ where the utility function is increasing in corn and decreasing in labor. The overall feasibility constraint is

$$\sum_{i=1}^{n} y_i = \lambda(\sum_{i=1}^{n} x_i)$$

The only difference with the previous example is that both inputs and outputs can be divided arbitrarily (in Example 3 the output is divided proportionally to input). Thus the feasible utility set $S(\lambda)$ is now

$$u \in S(\lambda) \Leftrightarrow \{ \exists (x_1,\ldots,x_n)(y_1,\ldots,y_n) : \sum_i y_i = \lambda(\sum_i x_i) \text{ and }$$
$$u_i = u_i(y_i,x_i) \text{ for all } i\}$$

Just as easily as in Example 3, one checks that property (8) holds, so that Theorem 1 applies.

When production exhibits increasing returns to scale, a natural Individual Rationality constraint consists of giving to individual agents free access to the technology: then $d_i(\lambda)$ is defined, again, by formula (11). In Moulin [1987a] it is shown that there is exactly one welfare egalitarian solution of the problem (d,S), namely the Constant Return Equivalent Solution given by (12). The intuitive argument is the same as in Example 3.
To summarize: in a production economy with one input and one output where the resource is the increasing returns to scale production function, there is a unique resource monotonic solution giving free access to the technology to every agent, namely the Constant Returns to Scale Solution. See Moulin [1987a] for a couple of minor topological assumtions; see also Moulin and Roemer [1986] and Moulin [1987b] for related results.

4. Welfare egalitarian solutions in two person problems

In the two agents case, we offer a characterization of the existence of a welfare egalitarian solution.

Consider Figure 2. Point a is the utility vector in $S(\lambda)$ where agent 2 gets his secure utility level $d_2(\lambda)$ and agent 1 pockets all the surplus left above $d(\lambda)$ in $S(\lambda)$. Similarly point b is the feasible utility vector in $S(\mu)$ where agent 2 pockets all the surplus left above $d(\mu)$ in $S(\mu)$, whereas agent 1 receives his bare secure utility level. In figure 2 point a is Northwest of point b so that a welfare egalitarian solution of problem (d, S) can not exist: indeed $f(\lambda)$ must be Northwest of a on $\partial S(\lambda)$ and $f(\mu)$ must be Southeast of b on $\partial S(\mu)$ therefore $f(\lambda)$ and $f(\mu)$ cannot be both on the same monotone utility path (both $f(\lambda) > f(\mu)$ and $f(\mu) > f(\lambda)$ are impossible).

The above argument uncovers a necessary condition for the existence of a welfare egalitarian solution. It turns out that this condition is sufficient as well.

Notation: Given a monotonic surplus-sharing problem (d,S) we denote by $a_i(\lambda)$ the maximal utility for agent i in $S(\lambda)$ compatible with Individual Rationality:

$$a_1(\lambda) = \max\{u_1/(u_1,d_2(\lambda)) \in S(\lambda)\} \tag{13}$$

(and a symmetrical definition for $a_2(\lambda)$). Thus the utility vectors $(a_1(\lambda),d_2(\lambda))$ (point a in Figure 2) and $(d_1(\lambda),a_2(\lambda))$ (point b) are weakly Pareto optimal in $S(\lambda)$.

Theorem 3

Given is a monotonic two person surplus-sharing problem $\Phi = (d,S)$. The two following statements are equivalent.

i) $WE_w(\Phi) \neq \phi$: there exists a welfare egalitarian weak solution of Φ

ii) for all $\lambda, \mu \varepsilon \Lambda$: $d_2(\lambda) < a_2(\mu)$ and/or $d_1(\mu) < a_1(\lambda)$ (14)

The proof is given in the Appendix. We illustrate the Theorem by one "negative" and one "positive" example.

Example 5 Cake Division

Two agents must share a vector of commodities $\lambda \varepsilon R_+^p$ without borrowing. Denote by $u_i(z_i)$ the nonnegative utility of agent i when he receives the nonnegative share z_i. Thus the feasible set is given by

$$S(\lambda) = \{u \varepsilon R_+^2 / \exists\ z_1, z_2 \varepsilon R_+^p : z_1 + z_2 = \lambda \text{ and } u_i = u_i(z_i) \text{ for } i = 1,2\}$$

In this problem the most popular secure utility level is the utility from consuming one half of the total commodity bundle (see Thomson and Varian [1986]):

$$d_i(\lambda) = u_i(\lambda/2) \quad \text{for } i = 1,2 \text{ and all } \lambda \varepsilon\ R_+^p$$

This IR requirement means that each agent can claim half of the commodities: an allocation mechanism must take advantage of the opportunities to trade from the initial allocation $(\lambda/2, \lambda/2)$.

The pair (d,S) constitutes a monotonic surplus-sharing problem for the partial ordering of R_+^p : when there is more resources to distribute there are more opportunities to trade, but the IR constraint is more demanding as well. The existence of a monotonic distribution of the cake is quite appealing since RM reflects the common property of the cake while IM offers to individuals the protection of the claim to their fair share of the cake. In Moulin and Thomson [1987] it was shown that for certain preference profiles, no such monotonic solution exists. We show below, however, that for many familiar profiles (e.g. when each agent has Cobb-Douglas preferences) a resource monotonic solution does exist (see Example 6 and Lemma 1).

For the time being we show that a welfare egalitarian solution of the Cake Division never exists unless both agents have essentially identical preferences.

Suppose first that u_1 and u_2 represent the same preferences. Then the equal division of resources ($z_i = \lambda/2$ for $i = 1,2$) is Pareto optimal as well, and the only monotonic solution is

$$f(\lambda) = (u_1(\lambda/2),\ u_2(\lambda/2))$$

It is welfare egalitarian as well (since there is a monotonic bijection b of $\mathbb{R}$ such that $u_2 = b(u_1)$). Suppose now that u_1 and u_2 do not represent the same preferences. Then we can find a budget set B such that the corresponding demands z_1 and z_2 of the two agents are distinct (for the sake of simplicity we assume that both demands are unique on B, but this is not essential to the argument).

The situation is depicted on Figure 3. We have drawn the upper contour sets U_1 of agent 1 at z_1 and U_2 of agent 2 at z_2. Also $\overline{U}_i$ is the symmetrical image of U_i around the vector z_i.

We check that property (14) is violated for $\lambda = 2z_1$, $\mu = 2z_2$. Indeed definition (13) reads:

$$a_2(\mu) = \max\{u_2(y_2)/u_1(\mu - y_2) \geq u_1(\mu/2)\} = \max_{y_2 \in \overline{U}_1}\{u_2(y_2)\}$$

And by construction:

$$a_2(\mu) = \max_{\overline{U}_1} u_2 < u_2(z_2) = d_2(\lambda)$$

Similarly

$$a_1(\lambda) = \max\{u_1(y_1)/u_2(\lambda - y_1) > u_2(\lambda/2)\} = \max_{\bar{U}_2} u_1 < u_1(z_1) = d_1(\mu)$$

Thus (14) is violated and there is no welfare egalitarian solution of the Cake Division problem.

Our second application of Theorem 3 is worth an independent statement:

<u>Corollary of Theorem 3</u>

Assume $\partial S(\lambda) = \partial_w S(\lambda)$ for all $\lambda \varepsilon \Lambda$. If the (two person) feasible set correspondence $S(\cdot)$ satisfies (8), then any monotonic surplus-sharing problem $\Phi = (d,S)$ has at least one welfare egalitarian solution.

The proof is given in the Appendix. Consider for instance the production economies described in Example 4 with two agents only. There property (8) holds on every domain Λ of production functions that is stable by the supremum operation. For instance Λ may consist of all nondecreasing production functions or all convex production functions such that $f(0) = 0$, or all increasing returns to scale production functions, or all decreasing returns to scale production functions; but Λ cannot be the set of all concave production functions (see Moulin and Roemer [1986] for a more detailed discussion).

If Λ is supremum stable the Corollary says that for <u>any</u> choice of the Individual Rationality constraint (such that the secure utility vector $d(\lambda)$ is feasible and monotonic w.r.t. λ) it is possible to construct a welfare egalitarian solution. Moreover, by Theorem 1, every monotonic solution of the corresponding surplus-sharing problem must be welfare egalitarian.

<u>5. Existence of monotonic solutions: a sufficient condition</u>

Consider a monotonic surplus-sharing problem $\Phi = (d,S)$ (Definition 1).

Pick an agent $i \varepsilon \{1,\ldots,n\}$ and denote by $a_i(\lambda)$ the maximal feasible utility level for agent i given the Individual Rationality constraint

$$a_i(\lambda) = \max\{u_i/(u_i, d_{-i}(\lambda)) \ \varepsilon \ S(\lambda)\} \tag{15}$$

Define the i-takes-all solution as the following mapping f^i:

$$f_i^i(\lambda) = a_i(\lambda) \ ; \ f_j^i(\lambda) = d_j(\lambda) \quad \text{all } j \neq i$$

In general f^i is weakly Pareto optimal and satisfies Individual Rationality (if $a_i(\lambda) = L$, then the set $\partial_w S(\lambda)$ defined by (2) may not contain $f^i(\lambda)$). Typically, then, f^i is a monotonic weak solution of Φ if and only if the function a_i is monotonic w.r.t.λ.

Admittedly the take-all solution f^i is not a fair monotonic solution of Φ. Yet in many examples, checking that a take all solution is monotonic is the quickest way to prove that Φ possesses monotonic solutions. This is particularly true in the Cake Division problem.

Example 6 Cake Division with Cobb-Douglas utilities

Suppose two agents have Cobb-Douglas utility function on R_+^p:

$$u_i(z_i) = \prod_{k=1}^{p} (z_{ik})^{\alpha_{ik}} \qquad i=1,\ldots,p$$

Agent i's take-all utility level $a_i(\lambda)$ is worth

$$a_i(\lambda) = \max\{u_i(z_i)/z_i \geq 0; \ u_j(\lambda - z_i) = u_j(\lambda/2)\}$$

An explicit computation is easily carried out. It turns out that the optimal solution of this program takes the form $z_{ik} = C_{ik}.\lambda_k$ where C_{ik} is a positive constant, whence $a_i(\lambda) = C \cdot u_i(\lambda)$ for some positive constant C.

Thus the take-all solutions of a cake division problem are always monotonic when the two agents have Cobb-Douglas utilities. In the two agents case, it is possible to derive also a sufficient condition.

Lemma 1

Consider a two-person cake division problem where the utility function u_1 is strictly monotonic and differentiable and where the cake λ varies within $\Lambda = \prod_{k=1}^{p} [0,A_k] \subseteq R_+^p$. Any division of the cake $\lambda\varepsilon\Lambda$ into two non negative shares z_1, z_2 is feasible.

Suppose that on every indifference curve of u_1, the marginal utility of each good does not double:

$$\text{for all } k = 1,\ldots,p, \text{ for all } x,y\varepsilon\Lambda, \{u_1(x) = u_1(y)\} \Rightarrow u_{1k}(x) < 2u_{1k}(y) \quad (16)$$

Then the 2-takes-all solution of the cake division is monotonic.

The proof is given in the Appendix. Lemma 1 implies for instance that take all solutions are monotonic when preferences are linear. Also, if the maximal cake $(A_1,\ldots A_p)$ is small enough, condition (16) holds true because u_1 is differentiable at the origin, so that take-all solutions are then monotonic. Thus, although the Cake Division problem has no welfare egalitarian solution (Example 5) it possesses frequently some monotonic solutions.

Remark 2

Condition (16) is not necessary for the 2-takes-all solution to be monotonic, as shown by the case of Cobb-Douglas preferences (Example 6): check that with $\Lambda = R_+^p$, on a Cobb-Douglas indifference curve the ratio of marginal utilities is not bounded above.

However, in a local sense, one can derive a necessary condition for the monotonicity of the function $a_2(\lambda) = \max\{u_2(z_2)/u_1(\lambda-z_2) = u_1(\lambda/2)\}$. Assuming that

$p = 2$ and that this program has a smooth solution $z_2(\lambda)$, it turns out that $a_2(\lambda)$ is monotonic in λ for every utility function u_2, if and only if:

$$u_{1k}(\lambda/2) \leq 2u_{1k}(\lambda - z_2(\lambda)) \text{ for all } \lambda$$

(we omit the tedious computations for brevity).

6. Existence of monotonic solutions: a necessary (and sometimes, sufficient) condition for two person problems

Lemma 2

Consider a two person monotonic surplus-sharing problem $\Phi = (d,S)$. If it possesses at least one monotonic weak solution ($M_w(\Phi) \neq \phi$) we have:

$$(a_1(\lambda),\ a_2(\mu)) \text{ is not an interior point of } S(\lambda \wedge \mu) \tag{17}$$

(where $a_1(\lambda)$, $a_2(\mu)$ are the take-all utilities defined by (13)).

Condition (17) says that the utility vector $u = (a_1(\lambda),\ a_2(\mu))$ is either unfeasible at $\lambda \wedge \mu$ ($u \notin S(\lambda \wedge \mu)$) or is a boundary point of $S(\lambda \wedge \mu)$ ($u \in \partial_w S(\lambda \wedge \mu)$: u is a weak Pareto optimum of $S(\lambda \wedge \mu)$ - see Lemma 5 in Appendix). Condition (17) is a weaker property than property (14). Indeed (14) implies that Φ had at least one welfare egalitarian weak solution, hence at least one monotonic weak solution. As a matter of Exercise, one can check directly that (14) implies (17).

Proof of Lemma 2

Let f be a weakly monotonic solution of Φ. By Individual Rationality of f we have

$$f_1(\lambda) \leq a_1(\lambda) \text{ for all } \lambda;\ f_2(\mu) \leq a_2(\mu) \text{ for all } \mu$$

Hence by monotonicity of f:

$$f_1(\lambda \wedge \mu) \leq f_1(\lambda) \leq a_1(\lambda);\ f_2(\lambda \wedge \mu) \leq f_2(\mu) \leq a_2(\mu)$$

Thus if $(a_1(\lambda), a_2(\mu))$ is an interior point of $S(\lambda \wedge \mu)$, so is $f(\lambda \wedge \mu)$, in contradition of weak Pareto optimality: $f(\lambda \wedge \mu) \in \partial_w S(\lambda \wedge \mu)$.

Example 7 Cake Division

With well behaved preferences (i.e. monotonic, convex) the condition (17) may fail for the cake division problem. This is, in essence, the Theorem 1 in Moulin and Thomson [1987]. Consider the profile depicted in Figure 4 where both agents have Leontieff (fixed proportions) preferences (Moulin and Thomson [1987] give an example with smooth preferences as well). Suppose the cake is the vector of commodities $\lambda = 2B$. Then by Individual Rationality, agent 1 can get at most the vector b so that $a_1(\lambda) = u_1(b)$. Similarly if the cake is the vector $\mu = 2C$, agent 1 can get at most c, so that $a_2(\mu) = u_2(c)$. It should be clear from Figure 4 that the cake $2A = \lambda \quad \mu$ can be divided between our two agents so as to give them strictly more than $a_1(\lambda)$ and $a_2(\mu)$ respectively.

The necessary condition (17) turns out to be also sufficient in an important class of two-person surplus-sharing problems. To introduce this class, we use a functional representation of the feasible utility sets $S(\lambda)$.

Definition 5

Given a two-person monotonic feasible set correspondence S from Λ into $\sum(2)$, we define two functions α, β as follows:

for all $\lambda \in \Lambda$, $I_1(\lambda) = \{u_1 / \exists\, u_2 \in [0,L] : (u_1,u_2) \in S(\lambda)\}$.

For all $u_1 \in I_1(\lambda)$ define $\alpha(u_1,\lambda) = \max\{u_2 / u_2 \in [0,L],\ (u_1,u_2) \in S(\lambda)\}$

for all $\lambda \in \Lambda$, $I_2(\lambda) = \{u_2 / \exists\, u_1 \in [0,L] : (u_1,u_2) \in S(\lambda)\}$.

For all $u_2 \in I_2(\lambda)$ define $\beta(u_2,\lambda) = \max\{u_1 / u_1 \in [0,L],\ (u_1,u_2) \in S(\lambda)\}$

Observe that α (resp. β) is increasing in λ and decreasing in u_1 (resp. u_2). As $S(\lambda)$ is comprehensive and closed, it is represented by these functions:

for all $u \in [0,L]^2$: $u \in S(\lambda) \Leftrightarrow u_2 < \alpha(u_1,\lambda) \Leftrightarrow u_1 < \beta(u_2,\lambda)$

Notice also that $a_1(\lambda) = \beta(d_2(\lambda),\lambda)$, $a_2(\lambda) = \alpha(d_1(\lambda),\lambda)$.

When the Pareto set and the weak Pareto sets of $S(\lambda)$ coincide, we can say even more:

Lemma 3

Suppose $\partial S(\lambda) = \partial_w S(\lambda)$. Then we have, for all $u \in [0,L]^2$:

$$u \in \partial S(\lambda) \Leftrightarrow u_1 = \beta(u_2,\lambda) \Leftrightarrow u_2 = \alpha(u_1,\lambda)$$

Moreover $\alpha(.,\lambda)$ (resp. $\beta(\cdot,\lambda)$) is a decreasing bijection from $J_1(\lambda) = \beta(I_2(\lambda),\lambda)$ into $J_2(\lambda) = \alpha(I_1(\lambda), \lambda)$ (resp. from $J_2(\lambda)$ into $J_1(\lambda)$) and these two bijections are inverse to each other.

The straightforward proof is omitted.

Definition 6

Let S be a two person monotonic feasible set correspondence from Λ into $\sum(2)$. Assume $\partial S(\lambda) = \partial_w S(\lambda)$ for all $\lambda \in \Lambda$. We say that S is supermodular if, for all λ, $\mu \in \Lambda$ it satisfies one of the three equivalent properties

i) for all $u_2 \in J_2(\lambda \wedge \mu)$: $\alpha(\beta(u_2,\mu),\lambda \vee \mu) > \alpha(\beta(u_2,\lambda \wedge \mu), \lambda)$

ii) for all $u_1 \in J_1(\lambda \wedge \mu)$: $\beta(\alpha(u_1,\lambda),\lambda \vee \mu) > \beta(\alpha(u_1,\lambda \wedge \mu), \mu)$

iii) for all $u, v \in [0,L]^2$:

$$\{u \in \partial S(\lambda \wedge \mu) \text{ and } (u_1,v_2) \in S(\lambda) \text{ and } (v_1,u_2) \in S(\mu)\} \Rightarrow \{v \in S(\lambda \vee \mu)\}$$

Property iii is illustrated by Figure 5. To check that it is equivalent to property i, invoke Lemma 3

$$u \in \partial S(\lambda \wedge \mu) \Leftrightarrow u_1 = \beta(u_2,\lambda \wedge \mu)$$

$$(u_1,v_2) \in S(\lambda) \Leftrightarrow v_2 < \alpha(u_1,\lambda)$$

$$(v_1,u_2) \in S(\mu) \Leftrightarrow v_1 < \beta(u_2,\lambda)$$

thus $v \in S(\lambda \quad \mu)$ easily implies property i).

Since $\beta(\cdot,\lambda \wedge \mu)$ is a bijection from $J_2(\lambda \wedge \mu)$ into $J_1(\lambda \wedge \mu)$ (Lemma 3) we can change the variable in property i) from u_2 to $u_1 = \beta(u_2,\lambda \wedge \mu)$. This in turn, yields property ii).

The terminology of supermodularity is justified by the <u>transferable utility</u> case where $L = +\infty$ and the feasibility constraint merely sets an upperbound on the sum of utilities. Let σ be a mapping from $[0,+\infty]$ into itself. We associate with σ the (transferable utility) feasible set correspondence:

$$S(\lambda) = \{(u_1,u_2) \in R_+^2 \,/\, u_1 + u_2 \leq \sigma(\lambda)\} \tag{18}$$

If σ is monotonic w.r.t. λ, so is $S(\lambda)$. For a TU feasible set like (18), we always have $\partial S(\lambda) = \partial_w S(\lambda)$. The corresponding functions α, β are simply:

$$\alpha(u_1,\lambda) = \sigma(\lambda)-u_1, \quad \text{all } u_1 \in I_1(\lambda) = [0,\sigma(\lambda)]$$

$$\beta(u_2,\lambda) = \sigma(\lambda)-u_2, \text{ all } u_2 \in I_2(\lambda) = I_1(\lambda)$$

Also $J_1(\lambda) = J_2(\lambda) = I(\lambda)$. Write now the supermodularity property of the mapping S by means of ii

$$\text{for all } u_1 \in [0,\sigma(\lambda)] : \sigma(\lambda \vee \mu) - (\sigma(\lambda)-u_1) \leq \sigma(\mu) - (\sigma(\lambda \wedge \mu)-u_1)$$

or, equivalently

$$\sigma(\lambda) + \sigma(\mu) \leq \sigma(\lambda \wedge \mu) + \sigma(\lambda \vee \mu) \tag{19}$$

The above inequality is the usual definition of a supermodular function on a lattice (see e.g. Topkis [1978]).

Theorem 4

Given is a two-person monotonic surplus-sharing problem $\Phi = (d,S)$ such that for all $\lambda \varepsilon \Lambda$ we have $\partial S(\lambda) = \partial_w S(\lambda)$.

Suppose that the correspondence S is supermodular (Definition 6). Then Φ has at least one monotonic solution ($M(\Phi) \neq \phi$) if and only if property (17) holds.

The proof is given in the Appendix.

As our first application, consider a transferable utility problem (d,σ) (the feasible utility set is defined by (18)). Then property (17) has a very simple expression, since $a_1(\lambda) = \sigma(\lambda) - d_2(\lambda)$, $a_2(\mu) = \sigma(\mu) - d_1(\mu)$:

$$(\sigma(\lambda) - d_1(\lambda)) + (\sigma(\mu) - d_2(\mu)) > \sigma(\lambda \wedge \mu) \iff$$

$$d_1(\lambda) + d_2(\mu) < \sigma(\lambda) + \sigma(\lambda) - \sigma(\lambda \wedge \mu) \text{ for all } \lambda,\mu \varepsilon \Lambda \qquad (20)$$

Theorem 4 now reads: if the transferable utility problem (d,σ) is supermodular (property (19)), then a monotonic solution exists if and only if property (20) holds true.

Our next example provides an illustration.

Example 8 Remuneration of inputs

Two agents can jointly produce some output y via a production function $y = h(\lambda_1,\lambda_2)$ where λ_i is agent i's input. Agent i alone can produce the amount $d_i(\lambda_i)$ of output, by using his own technology d_i. We assume that inputs vary within some interval $[\underline{\lambda},\overline{\lambda}] \subseteq R_+$ such that cooperation always pays, namely:

$$\text{for all } \lambda_1,\lambda_2 \ \varepsilon \ [\underline{\lambda},\overline{\lambda}] : d_1(\lambda_1) + d_2(\lambda_2) < \sigma(\lambda_1,\lambda_2) \qquad (21)$$

Just like in Example 4, we think of the production function σ as jointly owned by the agents. Unlike Example 4, it is kept constant in this example.

Instead we look at the inputs λ_1, λ_2 as the common property resource and explore the possibility of sharing monotonically the cooperative surplus $\{\sigma(\lambda_1,\lambda_2) - d_1(\lambda_1) - d_2(\lambda_2)\}$. This viewpoint makes good sense when the input measures some kind of nontransferable skill with which the agents are naturally endowed, and from the consumption of which they derive no utility. By looking for a monotonic solution we want to ensure that i) the opportunity to cooperate is fully exploited (Pareto optimality), ii) each agent receives at least his opportunity output (Individual Rationality) and iii) everyone benefits when someone's skill increases (Resource Monotonicity). Notice the tension between IR, conveying a right of every agent to use privately his input and RM, implied by the postulate that anyone's skill is common property as long as it is utilized in the public production process σ.

Formally we have a transferable utility problem (d,S) where the resource parameter is $\lambda - (\lambda_1,\lambda_2)$ varying in $\Lambda = [\underline{\lambda},\overline{\lambda}]^2$ and the feasible utility set is

$$S(\lambda) = \{u \in [0,L]^2 / u_1 + u_2 < \sigma(\lambda_1,\lambda_2)\}$$

(where we set $L = \sigma(\underline{\lambda},\overline{\lambda})$).

This is a monotonic problem (Definition 1) because the production functions σ, d_1 and d_2 are input monotonic.

Consider the supermodularity property (19). If we choose $\lambda_1 < \mu_1$, $\mu_2 < \lambda_2$, it implies:

$$\sigma(\lambda_1,\lambda_2) + \sigma(\mu_1,\mu_2) < \sigma(\lambda_1,\mu_2) - \sigma(\mu_1,\lambda_2)$$

$$\iff \sigma(\mu_1,\mu_2) - \sigma(\lambda_1,\mu_2) < \sigma(\mu_1,\lambda_2) - \sigma(\lambda_1,\lambda_2)$$

This is the familiar condition of input complementarity. When σ is twice differentiable it amounts to $\sigma_{12} > 0$: the marginal product of one input increases as the utilization of the other input increases. One checks easily that input complementarity is equivalent to supermodularity (19).

Assuming input complementarity in the joint production function σ, we know now that a monotonic solution exists if and only if (20) holds, namely:

$$d_1(\lambda_1) + d_2(\mu_2) \leq \sigma(\lambda_1,\lambda_2) + \sigma(\mu_1,\mu_2) - \sigma(\lambda_1 \wedge \mu_1, \lambda_2 \wedge \mu_2), \text{ all } \lambda, \mu \varepsilon \Lambda$$

This inequality can be considerably simplified. Fix λ_1, and μ_2 and compute

$$\inf_{\underline{\lambda} \leq \mu_1,\lambda_2 \leq \overline{\lambda}} \{\sigma(\lambda_1,\lambda_2) + \sigma(\mu_1,\mu_2) - \sigma(\lambda_1 \wedge \mu_1, \lambda_2 \wedge \mu_2)\} =$$

$$\inf_{\underline{\lambda} \leq \mu_1 \leq \lambda_1, \underline{\lambda} \leq \lambda_2 \leq \mu_2} \{\sigma(\lambda_1,\lambda_2) + \sigma(\mu_1,\mu_2) - \sigma(\mu_1,\lambda_2)\} = \sigma(\lambda_1,\underline{\lambda}) + \sigma(\underline{\lambda},\mu_2) - \sigma(\underline{\lambda},\underline{\lambda})$$

(the latter equality from input complementarity). Thus a monotonic solution exists if and only if

$$d_1(\lambda_1) + d_2(\mu_2) \leq \sigma(\lambda_1,\underline{\lambda}) + \sigma(\underline{\lambda},\mu_2) - \sigma(\underline{\lambda},\underline{\lambda}), \text{ all } \lambda_1,\mu_2 \ \varepsilon \ [\underline{\lambda},\overline{\lambda}] \quad (22)$$

To analyze this condition, distinguish two cases.

<u>Case i)</u>: $\sigma(\underline{\lambda},\underline{\lambda}) = 0$: at the minimal input level, no output is produced (for instance $\underline{\lambda} = 0$). Then the feasibility condition (21) implies $d_i(\underline{\lambda}) = 0$ (take $\lambda_1 = \lambda_2 = 0$), whence $d_1(\lambda_1) \leq \sigma(\lambda_1,\underline{\lambda})$ (take $\lambda_2 = \underline{\lambda}$) and similarly $d_2(\lambda_2) \leq \sigma(\underline{\lambda},\lambda_2)$. Therefore condition (22) always holds and existence of monotonic solutions is warranted.

Actually a monotonic solution is easily constructed:

$$f_1(\lambda) = \sigma(\lambda_1,\underline{\lambda}); \; f_2(\lambda) = \sigma(\lambda_1,\lambda_2) - \sigma(\lambda_1,\underline{\lambda})$$

Here agent 1 gets the output that he could produce by having free access to the technology σ, and agent 2 takes all the surplus. Monotonicity of f_2 w.r.t. λ

follows directly from input complementarity.

Case ii): $\sigma(\underline{\lambda},\underline{\lambda}) > 0$: at the minimal imput level, there is some output to be shared. Then it is easy to construct examples where condition (22) is violated: despite input complementarity and the cooperative opportunity (21), there is no monotonic surplus-sharing solution.

The idea in the example is that the cooperative surplus is large when input levels are low (due to large cooperative premium $\sigma(\underline{\lambda},\underline{\lambda}) - d_1(\underline{\lambda}) - d_2(\underline{\lambda})$) but diminishes when inputs rise, so as to violate (22) for nearly maximal input level

For instance,

$$\underline{\lambda} = 0,\ \overline{\lambda} = 1\ ;\ d_i(\lambda_i) = \lambda_i\ ;\ \sigma(\lambda_1,\lambda_2) = 1 + (.36)(\lambda_1+\lambda_2)^2$$

Check that σ satisfies input complementarity ($\sigma_{12} = .72$) and condition (21) but violates (22) for $\lambda_1 = \lambda_2 > .7$.

Example 9 Cake Division:

Consider the Cake Division problem between two agents with utility functions u_1, u_2. Assume that the resource λ varies in the whole positive orthant $\Lambda = R_+^p$ (this assumption entails no loss of generality).

We analyze the supermodularity of the feasible utility correspondence

$$S(\lambda) = \{u\varepsilon R_+^2 / \exists\, z_1, z_2 \varepsilon R_+^p,\ z_1 + z_2 = \lambda \text{ and } u_i = u_i(z_i),\ i = 1,2\} \quad (23)$$

We will use the following equivalent formulation of property i in Definition 6. For all λ, $\mu\varepsilon\Lambda$ we have

$$\text{for all } u_1 \ \varepsilon\ J_1(\lambda \wedge \mu),\ u_1' \ \varepsilon\ J_1(\mu) : \{\alpha(u_1,\lambda\wedge\mu) = \alpha(u_1',\mu)\} \Rightarrow$$
$$\{\alpha(u_1,\lambda) \leq \alpha(u_1',\lambda\vee\mu)\} \quad (24)$$

For simplicity we assume first that p = 2 (two commodities). Check that for any λ, $\mu\varepsilon\Lambda$ such that $\lambda \leq \mu$ the above property is trivially satisfied. The only interesting case is thus when λ and μ are not comparable.

Fix some vector $w = (w_1,w_2)$ in R_+^2 and two vectors $\delta = (x_1,0)$, $\varepsilon = (0,x_2)$, with $x_1,x_2 > 0$. We set $\lambda = w+\delta$, $\mu = w+\varepsilon$ so that $\lambda \wedge \mu = w$, $\lambda \vee \mu = w+\delta+\varepsilon$. Then property (24) can be rewritten as:

$$\text{for all } u_1 \varepsilon J_1(w),\ u_1' \varepsilon J_1(w+\varepsilon)$$

$$\max\{u_2(z_2)/u_1(w-z_2) \geq u_1\} = \max\{u_2(z_2)/u_1(w+\varepsilon-z_2) \geq u_1'\} \tag{25}$$

$$\Rightarrow \max\{u_2(z_2)/u_1(w+\delta-z_2) \geq u_1\} \leq \max\{u_2(z_2)/u_1(w+\delta+\varepsilon-z_2) \geq u_1'\}$$

<u>Lemma 4</u>

Suppose that both agents have monotonic convex preferences and that no good is inferior for any agent:

$$\text{for any two goods } k,\ k' \quad (u_i)_k \cdot (u_i)_{kk'} - (u_i)_{k'} \cdot (u_i)_{k^2} \geq 0 \qquad \text{for } i = 1,2 \tag{26}$$

Then the feasible utility correspondence S ((23)) is supermodular.

When preferences satisfy the above assumptions, Theorem 4 says that a monotonic solution of the Cake Division problem exists if and only if condition (17) holds true. Let us formulate (17) for this specific problem.

Pick a vector $w \varepsilon R_+^p$ and two non negative vectors δ, ε in R_+^p such that

$$\text{for all } k = 1,\ldots,p \qquad \varepsilon_k \cdot \delta_k = 0$$

Then (17) must hold with $\lambda = w+\delta$, $\mu = w+\varepsilon$, $\lambda \wedge \mu = w$:

$$a_1 = \max\{u_1(z_1)/u_2(w+\delta-z_1) \geq u_2(w+\delta/2)\},$$

$$a_2 = \max\{u_2(z_2)/u_1(w+\varepsilon-z_2) \geq u_1(w+\delta/2)\} \Rightarrow \tag{27}$$

$$a_2 \geq \max\{u_2(z_2)/u_1(w-z_2) \geq a_1\}$$

Admittedly, this property is not easy to check by explicit computations. It says that existence of a monotonic cake division is guaranteed if we can avoid the configuration of Example 7, or more generally that of Figure 1 in Moulin and Thomson [1987].

7. Deficit sharing

Sometimes cooperation is needed to share a deficit rather than a surplus. A good example is the bankruptcy model discussed by O'Neill [1982], Aumann and Maschler [1985], and Young [1986]. Another example is collusion in oligopoly, as discussed in the introduction.

Formally a deficit-sharing situation is a pair (c,S) where S is the set of feasible utility sets and c is the maximal utility vector (or cap vector) giving an upper bound c_i for agent i's utility vector. We make the same assumptions on $S(\cdot)$ and we assume that c is outside S or is in the Pareto set ∂S.

A monotonic deficit-sharing problem $\Psi = (c,S)$ is defined similarly: both $S(\lambda)$ and the cap vector $c(\lambda)$ vary monotonically w.r.t. the resource parameter λ. A monotonic solution (resp. weak solution) of Ψ is a monotonic selection f of the Pareto correspondence $\partial_w S$ that is bounded above by the cap vector:

$$f(\lambda) \leq c(\lambda) \quad \text{for all } \lambda \tag{28}$$

This inequality replaces Individual Rationality. All the results for surplus-sharing problems can be extended or adapted to deficit-sharing, with identical proof techniques.

Consider first Theorems 1 and 2 giving sufficient conditions for all monotonic solutions to be actually welfare egalitarian. These results extend word for word to deficit-sharing. Indeed the whole point is to show that a monotonic selection of ∂S must be path monotone, quite independently of additional constraints on the selection, such as Individual Rationality or the upper bound (28). See the Appendix.

For instance, consider the production economies (one input, one output) with decreasing returns to scale (as opposed to Example 4 where returns to scale are increasing). The utility vector formed by giving to each agent free access to the technology is now unfeasible - or barely feasible - so it can be taken as the cap of agents' utility: since joint utilization of the production function is less efficient, every agent bears some of the cost of this disexternality. This deficit-sharing problem is discussed in Moulin [1987a]. As the domain of DRS production function is stable by the supremum operation, we can still apply Theorem 1. It turns out that the Constant Returns to Scale solution is the only one possible; it is actually a solution if the agents have convex preferences (see Section 5 of Moulin [1987a]).

We describe now in more detail two similar deficit-sharing problems.

Example 10 Extraction of common property resources

Consider the model of Example 3, where we assume instead that the catch function λ has decreasing returns to scale: $\lambda(x)/x$ is non increasing in x. This says that the other agents' fishing effort is a negative externality, a realistic assumption when fishing is intense. As in Example 3 our agents share the production function λ. Unlike Example 3 the utility levels that they achieve when given free access to the technology are not together feasible. Define:

$$c_i(\lambda) = \max_{x_i \geq 0} u_i(\lambda(x_i), x_i),$$

namely the utility available to agent i if he is fishing alone. It is natural to use $c_i(\lambda)$ as a cap on individual utility levels: the disexternality creates the deficit to the provision of which everyone must contribute.

As in Example 3, Theorem 1 applies: every monotonic deficit-sharing solution is welfare egalitarian. Again, there is at most one such solution.

namely the Constant Returns Equivalent solution defined exactly as before. The point is that when production has constant returns to scale ($\lambda = \lambda_a$) the cap vector $c(\lambda_a)$ is Pareto optimal; the rest of the proof follows similarly. To prove that the CRE solution satisfies the constraint (28), we actually need to assume that the agents' preferences are convex (as in Section 5 of Moulin [1987a]). Without convex preferences, it is possible that the CRE solution violates (28), implying that our surplus-sharing problem has no monotonic solution.

Example 11 Quantity-setting oligopoly

The n agents are n oligopolists endowed with respective cost functions $\gamma_i(x_i)$, for $i = 1,\ldots,n$. The common property resource is the inverse demand function $\mu(x)$ giving the price $p = \mu(x)$ that clears the total supply $x = \sum_{i=1}^{n} x_i$. Firm i's utility is its profit $u_i = x_i \cdot \mu(x) - \gamma_i(x_i)$.

Our n firms collude: they wish to pick an outcome $x_1,\ldots,x_n$ on the profit possibility frontier (we exclude side payments, although this assumption is not crucial). Resource monotonicity means that as the demand increases, the profit of no firm should go down. We assume that Λ is the domain of all non increasing functions vanishing beyond a certain maximal output level $\overline{x}$.

A natural upper bound on any firm's profit is the monopoly profit $c_i(\mu)$:

$$c_i(\mu) = \max_{x_i \geq 0}\{x_i \cdot \mu(x_i) - \gamma_i(x_i)\}$$

The idea is clear: without competition, firm i would not do better than $c_i(\mu)$ anyway; everyone must make a concession because competition cannot be avoided.

Formally this deficit-sharing model is a particular case of that in Example 10, where the utility function is taken to be

$$u_i(y_i,x_i) = y_i-\gamma_i(x_i)$$

and the production function λ is just the revenue function:

$$\lambda(x) = x \cdot \mu(x)$$

so that firm i receives the share x_i/x of total revenue:

$$u_i(\frac{x_i}{x}\cdot\lambda(x),\ x_i) = x_i\cdot\mu(x) - \gamma_i(x_i)$$

The only difference between the two models (in Example 10 and in the current model) bears on the domains: if Λ contains essentially any decreasing function, then λ defined in (29) is not necessarily increasing, so it cannot be interpreted as a production function.

Notwithstanding this discrepancy, the essential arguments of Example 10 carry over to the oligopoly model. Firstly the profit possibility set $S(\mu)$ satisfies (8) hence every monotonic deficit-sharing has to be welfare egalitarian. Secondly the only possible welfare egalitarian solution follows the monotone profit path:

$$a \to c(\mu_a) \text{ where } \mu_a(x) = a, \text{ all } x$$

Indeed if the demand function is μ_a our oligopolists face a competitive demand and each one can achieve its monopoly profit. Thirdly one must check that the following solution

for every decreasing μ, $f^*(\mu) = c(\mu_a*)$ where a^*
is the largest number such that $c(\mu_a*) \in S(\lambda)$

satisfies the upper bound condition $f_i^*(\mu) \leq c_i(\mu)$ for all agent i and all demand function μ. This, again, holds true if the cost function c_i are all convex.

We review quickly the adaptation of Theorem 3 and 4 to deficit-sharing problems. For the existence of welfare egalitarian solutions, define:

$$b_1(\lambda) = \max\{u_1/(u_1,c_2(\lambda))\varepsilon S(\lambda)\},\ b_2(\lambda) = \max\{u_2/(c_1(\lambda),u_2)\varepsilon S(\lambda)\}$$

Then the necessary and sufficient condition of existence of a welfare egalitarian weak solution is

$$\text{for all } \lambda,\ \mu\varepsilon\Lambda\text{:}\quad b_2(\mu) < c_2(\lambda) \text{ and/or } b_1(\lambda) < c_1(\mu)$$

Next the Corollary of Theorem 3 becomes: if $\partial S = \partial_w S$ and the two person feasible correspondence S satisfies (9) (instead of (8)), then any monotonic deficit-sharing problem (c,S) has at least one welfare egalitarian solution.

For the existence of monotonic solutions, the necessary condition of Lemma 2 becomes:

$$\text{for all } \lambda,\mu\varepsilon\Lambda\quad (b_1(\lambda),\ b_2(\mu))\ \varepsilon\ S(\lambda\vee\mu) \tag{30}$$

In Theorem 4, supermodularity of S must be replaced by submodularity. The correspondence S (such that $\partial S = \partial_w S$) is submodular if for all $\lambda,\mu\varepsilon\Lambda$ we have

$$\alpha(\beta(u_2,\mu),\lambda\wedge\mu) < \alpha(\beta(u_2,\lambda\vee\mu),\lambda)\quad \text{all } u_2\varepsilon J_2(\lambda\wedge\mu)$$

In the transferable utility case, this reads

$$\sigma(\lambda\wedge\mu) + \sigma(\lambda\vee\mu) < \sigma(\lambda) + \sigma(\mu)\quad \text{all } \lambda,\mu$$

Finally Theorem 4 is adapted as follows. Suppose the correspondence S satisfies $\partial S = \partial_w S$ and is submodular. Then a monotonic two-person deficit-sharing problem has at least one monotonic solution if and only if property (30) holds.

8. Concluding comments

So far only Theorems 1 and 2 are applicable to an arbitrary number of agents.

For two person problems, we can give an completely general characterization of existence of monotonic solutions based upon a constructive proof: this result, despite its technical interest has yet to find relevant applications, which explain why it is relegated at the end of the mathematical appendix (Theorem 5).

Beyond the existence result (is $M(\Phi)$ empty or not?), lies the deeper and more difficult question of analyzing the structure of the set $M(\Phi)$ of monotonic solutions. We illustrate the difficulty by stating two natural but unsolved questions.

For two person problems, one checks easily that $M(\Phi)$ must contain (at least when $\partial S(\lambda) = \partial_w S(\lambda)$ for all λ) a best solution for agent 1 and a best solution for agent 2. Indeed for any two solutions f, g of Φ, by picking for each λ his preferred element $f(\lambda)$ or $g(\lambda)$, agent 1 (resp. 2) defines another solution of Φ. Actually the solution best for a certain agent is used in the proof of Theorem 3 and of Theorem 4 (see Remark 4 in the Appendix).

Question 1: With three agents or more, can we still speak of the best solution for a certain agent?

Question 2: In a (two person) problem Φ can we find monotonic solutions that compromise arbitrarily between these two extremes? To put the question mathematically, is the set $M(\Phi)$ of solutions arc-connected?

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Appendix: Proofs

1. Proof of Theorems 1 and 2

We state first without proofs a few simple facts

Lemma 5

Let S be a closed, comprehensive and non empty subset of $[0,L]^n$. Then its weak Pareto frontier $\partial_w(S)$ (defined by (1)(2)) is its topological boundary.

Lemma 6

Let S be as above. Then the two following statements are equivalent:

i) $\partial S = \partial_w S$

ii) $\partial_w(S)$ has no flat part: $\forall\ u, v \varepsilon \partial_w(S)\ \{u \geqslant v\} \Rightarrow \{u = v\}$

Lemma 7

Let S be as above and γ be a monotone utility path (Definition 3). Then the set $I = \{t\ \varepsilon\ [0,L]/\gamma(t)\ \varepsilon\ \partial_w(S)\}$ is nonempty and closed. We define

$\gamma \cap \partial_w(S) = \gamma(t^*)$ where t^* is the largest element of I

If $\gamma([0,L])$ intersects the Pareto frontier $\partial(S)$, then the intersection coincides with $\gamma \cap \partial_w(S)$ and will be denoted $\gamma \cap \partial(S)$.

Lemma 8

The mapping $\gamma \cap \partial_w(S)$ is monotonic w.r.t.S:

$S \subsetneq S' \Rightarrow \gamma \cap \partial_w(S) \leqslant \gamma \cap \partial_w(S')$

The next Lemma is essential to the proof of Theorems 1, 2.

Lemma 9

Given are a correspondence $S(\cdot)$, associating with each $\lambda\varepsilon\Lambda$ a closed comprehensive subset $S(\lambda) \subseteq [0,L]^n$, and a selection $f(\cdot)$ of its Pareto set:

for all $\lambda\varepsilon\Lambda$, $f(\lambda) \;\varepsilon\; \partial S(\lambda)$

Then the two following statements are equivalent:

i) there exists a monotone utility path γ (Definition 3) such that:

for all $\lambda\varepsilon\Lambda$, $f(\lambda) = \gamma \cap \partial S(\lambda)$

ii) for all $\lambda,\mu\varepsilon\Lambda$, $\{f(\lambda) \leqslant f(\mu)\}$ and/or $\{f(\mu) \leqslant f(\lambda)\}$

Proof: Any two vectors in the range of γ can be ordered, whence i) implies ii). Conversely, assume that f satisfies ii). Then $f(\Lambda)$ is a chain, i.e. a completely ordered subset of $[0,L]^n$. By Zorn's lemma, $f(\Lambda)$ is contained in a maximal (w.r.t. inclusion) chain Γ of $[0,L]^n$.

Check now that a maximal chain Γ must be the image of a monotone utility path: for some such path γ (Definition 3), we have $\Gamma = \gamma([0,L])$. We prove this claim when Γ is finite ($L < +\infty$). The case $L = +\infty$ is left to the careful reader. Denote by $H(t)$ the following hyperplane:

$$H(t) = \{u\varepsilon[0,L]^n / \sum_{i=1}^{n} u_i = nt\} \quad \text{for } t\varepsilon[0,L]$$

Because Γ is a chain its intersection with $H(t)$ contains at most one point, for all t. Because Γ is maximal, it does intersect every set $H(t)$. Assume to the contrary, $\Gamma \cap H(t) = \phi$ for some t. Then we have

$$\sup\{u/u\varepsilon\Gamma, \sum_{i=1}^{n} u_i < nt\} = u_- \leqslant u_+ = \inf\{u/u\varepsilon\Gamma, \sum_{i=1}^{n} u_i > nt\}$$

We can pick $\bar{u}$ in $H(t)$ such that $u_- \leqslant \bar{u} \leqslant u_+$, therefore $\Gamma \cup \{\bar{u}\}$ is a chain containing Γ, contradicting the maximality of Γ.

Denote by $\gamma(t)$ the intersection $\Gamma \cap H(t)$, for all $t\varepsilon[0,L]$. It is now clear that γ is a monotone utility path (property ii holds because Γ is a chain) with image Γ: $\gamma([0,L]) = \Gamma$. By construction, for every λ, the vector $f(\lambda)$ is in $\Gamma \cap \partial S(\lambda)$, so by Lemma 7, $f(\lambda) = \gamma \cap \partial S(\lambda)$ as was to be proved.

Note that if we assume only that $f(\lambda)$ is weakly Pareto optimal in $S(\lambda)$, the implication i) => ii) still holds but its converse is no longer true.

Proof of Theorem 1

Since $WE(\Phi)$ is a subset of $M(\Phi)$ we only need to show the converse inclusion. Let f be a monotonic solution of problem Φ. We pick two parameters $\lambda,\mu\varepsilon\Lambda$, and compare $f(\lambda \vee \mu)$ with $f(\lambda)$ and $f(\mu)$. In view of (8), and the inclusion $\partial(A \cup B) \subset \partial(A) \cup \partial(B)$, we have:

$$f(\lambda \vee \mu) \in \partial S(\lambda \vee \mu) = \partial(S(\lambda) \cup S(\mu)) \subset \partial S(\lambda) \cup \partial S(\mu)$$

Suppose that $f(\lambda \vee \mu)$ belongs to $\partial S(\lambda)$. By the monotonicity of f, S and the definition of Pareto optimality we get:

$$\{f(\lambda) \leq f(\lambda \vee \mu) \text{ and } f(\lambda), f(\lambda \vee \mu) \in \partial S(\lambda)\} \Rightarrow \{f(\lambda) = f(\lambda \vee \mu)\}$$
$$\Rightarrow f(\mu) \leq f(\lambda)$$

Similarly if $f(\lambda \vee \mu)$ belongs to $\partial S(\mu)$ we obtain $f(\lambda) \leq f(\mu)$.

We have shown that f satisfies assumption ii) of Lemma 9. Hence there exists a monotone utility path γ satisfying (7), as was to be proved.

Proof of Theorem 2

Let f be a monotonic solution of problem Φ. We pick two paramters $\lambda,\mu\varepsilon\Lambda$ and compare $f(\lambda \wedge \mu)$ with $f(\lambda)$ and $f(\mu)$.

Lemma 7 implies that for any two closed comprehensive subsets A, B of $[0,L]^n$ we have:

$$\partial_w(A \cap B) \subset \partial_w(A) \cup \partial_w(B)$$

Combining this with assumption (9) we get

$$f(\lambda \wedge \mu) \in \partial_w S(\lambda \wedge \mu) = \partial_w(S(\lambda) \cap S(\mu)) \subset \partial_w S(\lambda) \cup \partial_w S(\mu)$$

suppose $f(\lambda \wedge \mu)$ belongs to $\partial_w S(\lambda)$. By (10), it is actually Pareto optimal as well; so that

$$\{f(\lambda \wedge \mu) \leq f(\lambda) \text{ and } f(\lambda \wedge \mu), f(\lambda) \in \partial S(\lambda)\} \Rightarrow \{f(\lambda \wedge \mu) = f(\lambda)\}$$
$$\Rightarrow \{f(\lambda) \leq f(\mu)\}$$

Similarly if $f(\lambda \wedge \mu)$ belongs to $\partial_w S(\mu)$ we obtain $f(\mu) \leq f(\lambda)$. Theorem 2 now follows from Lemma 9.

Remark 3

Under assumption (10) we can combine the two above proofs to state a (slightly) stronger result. Suppose that the mapping S satisfies:

$$\text{for all } \lambda,\mu \in \Lambda \ \{S(\lambda \vee \mu) = S(\lambda) \cup S(\mu)\} \text{ and/or } S(\lambda \wedge \mu) = S(\lambda) \cap S(\mu)\}$$

Then the conclusion of Theorems 1, 2 holds, namely $WE(\Phi) = M(\Phi)$.

2. Proof of Theorem 3

i) => ii)

Pick f in $WE_w(\Phi)$. Then for all λ and all $\{i,j\} = \{1,2\}$ we have:

$$\{f(\lambda) \in S(\lambda), f_i(\lambda) \geq d_i(\lambda)\} \Rightarrow f_j(\lambda) \leq a_j(\lambda)$$

Next for any pair $\lambda,\mu \in \Lambda$ the two vectors $f(\lambda)$ and $f(\mu)$ are ordered: say $f(\lambda) \leq f(\mu)$. Then:

$$d_2(\lambda) \leq f_2(\lambda) \leq f_2(\mu) \leq a_2(\mu)$$

Similarly $f(\mu) \leq f(\lambda)$ implies $d_1(\mu) \leq a_1(\lambda)$

To prove the converse implication ii) => i) we need a preliminary result. We say that a subset A of $[0,L]^2$ is comprehensive if $\{u\varepsilon A, v \leq u\}$ implies $v\varepsilon A$. We say that A is *-comprehensive if $\{u\varepsilon A, v_1 \geq u_1, v_2 \leq u_2\}$ implies $v\varepsilon A$.

Lemma 10

Let S be a closed, comprehensive nonempty subset of $[0,L]^2$ and D be a closed, *-comprehesive, nonempty subset of $[0,L]^2$. Suppose that two utility vectors x, y are such that

i) both x, y are weakly Pareto optimal in S ($x, y \varepsilon \partial_w(S)$) and

$y_1 \leq x_1$, $x_2 \leq y_2$

ii) x belongs to D and y is not an interior point of D

Then there exists a vector u such that

i) $y_1 \leq u_1 \leq x_1$, $x_2 \leq u_2 \leq y_2$

ii) u is weakly Pareto optimal in S: $u\varepsilon\partial_w(S)$

iii) u is a boundary point of D

Proof of Lemma 10

For every θ in the interval $[x_2-x_1, y_2-y_1]$, the straight line $\Delta(\theta) = \{u\varepsilon[0,L]^2/u_2-u_1 = \theta\}$ intersets $\partial_w(S)$ in at most one point (otherwise $\partial_w(S)$ contains two points u, v such that $u << v$). In fact $\Delta(\theta)$ intersects $\partial_w(S)$ in exactly one point. This is obvious if $\theta = x_2-x_1$ or $\theta = y_2-y_1$. If θ belongs to $]x_2-x_1, y_2-y_1[$, the smallest vector in $\Delta(\theta)$ belongs to S (because S is comprehensive) and the largest one is outside S, because u and v are in $\partial_w(S)$ (to check this, draw a figure and distinguish the cases $u_1 = L$, $u_1 < L$ as well as $v_2 = L$, $v_2 < L$).

We denote by $u(\theta)$ the unique intersection of $L(\theta)$ with $\partial_w(S)$ (See Figure 6). The mapping $\theta \to u(\theta)$ thus parametrizes the portion of $\partial_w(S)$ between x and y for θ varying in the interval $[x_2-x_1, y_2-y_1]$. Clearly $u_1(\theta)$ is nonincreasing in Θ while $u_2(\Theta)$ is nondecreasing. As $u_2(\Theta) - u_1(\Theta) = \Theta$ for all Θ, this implies that $u(\cdot)$ is continuous in Θ. Since $u(x_2-x_1)$ belongs to D while $u(y_2-y_1)$ is not interior to D, there must exist some Θ such that $u(\Theta)$ is a boundary point of D. The vector $u = u(\Theta)$ satisfies all three properties i)-iii). The proof of Lemma 10 is now complete.

We prove now the implication ii) => i) of Theorem 3. Given is a monotonic surplus-sharing problem Φ satisfying (14). We will show the existence of a chain Γ such that for all λ

$$\text{the intersection of } \partial_w S(\lambda) \text{ and } \Gamma \text{ is nonempty and contains a vector } u \text{ such that } d(\lambda) \leq u \tag{31}$$

Then by the argument used in the proof of Lemma 9 we extend the chain into a monotone utility path γ and define $f(\lambda) = \gamma \cap \partial_w S(\lambda)$, to be the desired welfare egalitarian weak solution of Φ.

For every $\lambda \in \Lambda$ we denote by $\beta^i(\lambda)$ the weakly Pareto optimal utility vector most advantageous to agent i:

$$\beta^1(\lambda) = (a_1(\lambda), \beta_2^1(\lambda)) \text{ where } \beta_2^1(\lambda) = d_2(\lambda) \text{ if } a_1(\lambda) < L$$
$$= \max\{u_2 / (L, u_2) \in S\} \text{ if } a_1(\lambda) = L$$

$$\beta^2(\lambda) = (\beta_1^2(\lambda), a_2(\lambda)) \text{ where } \beta_1^2(\lambda) = d_1(\lambda) \text{ if } a_2(\lambda) < L$$
$$= \max\{u_1 / (u_1, L) \in S\} \text{ if } a_2(\lambda) = L$$

One checks that $\beta^i(\lambda)$ always belongs to $\partial_w S(\lambda)$: See Figure 7.

Next we construct the desired chain Γ that will be most advantageous to agent 2. Define the following subset E

$$E = \{u \in [0,L]^2 / \exists\ \lambda \in \Lambda\ \{u_1 < \beta_1^2(\lambda),\ u_2 > \beta_2^2(\lambda)\} \text{ or } \{u_1 = \beta_1^2(\lambda) = 0,$$
$$u_2 > \beta_2^2(\lambda)\} \text{ or } \{u_1 < \beta_1^2(\lambda),\ u_2 = \beta_2^2(\lambda) = L\}\}$$

Clearly E is open and its complement $D = [0,L]^2 \setminus E$ is closed, *-comprehensive, and nonempty (it contains $(L,0)$). If E is empty then $\beta_2(\lambda) = (0,L)$ for all λ, so that the trivial chain $\Gamma = \{(0,L)\}$ satisfies (31). From now on, assume that E is nonempty, or equivalently that $(0,L) \varepsilon E$.

Fix a parameter $\lambda \varepsilon \Lambda$. We claim that $\beta^1(\lambda)$ belongs to D. Otherwise, it must belong to E so there exists $\mu \varepsilon \Lambda$ such that

$$a_1(\lambda) = \overset{1}{\beta_1}(\lambda) < \overset{2}{\beta_1}(\mu), \quad \overset{1}{\beta_2}(\lambda) > \overset{2}{\beta_2}(\mu) = a_2(\mu)$$

This implies $a_1(\lambda) < L$, $a_2(\mu) < L$, hence $\overset{1}{\beta_2}(\lambda) = d_2(\lambda)$, $\overset{2}{\beta_1}(\mu) = d_1(\mu)$. Therefore we have a contradiction of the assumption (14).

Next we claim that $\beta^2(\lambda)$ is not an interior point of D. Assume, to the contrary, that $\beta^2(\lambda)$ is interior to D. Firstly $\beta^2(\lambda) \neq (0,L)$ since we have assumed $(0,L) \varepsilon E$. Then distinguish three cases:

.if $\overset{2}{\beta_1}(\lambda) > 0$ and $\overset{2}{\beta_2}(\lambda) < L$, then we can find $u \varepsilon D$ such that $u_1 < \overset{2}{\beta_1}(\lambda)$ and $u_2 > \overset{2}{\beta_2}(\lambda)$, implying $u \varepsilon E$, a contradiction.

.if $\overset{2}{\beta_1}(\lambda) = 0$ and $\overset{2}{\beta_2}(\lambda) < L$, then we can find $u \varepsilon D$ such that $u_1 = 0$ and $u_2 > \overset{2}{\beta_2}(\lambda)$, implying again $u \varepsilon E$, a contradiction.

.if $\overset{2}{\beta_1}(\lambda) > 0$ and $\overset{2}{\beta_2}(\lambda) = L$ the argument is similar.

Summarizing, we have shown that $\beta^1(\lambda)$ belongs to D and $\beta^2(\lambda)$ is not interior to D. By construction they are both in $\partial_w S(\lambda)$ and $\overset{2}{\beta_1}(\lambda) < \overset{1}{\beta_1}(\lambda)$, $\overset{1}{\beta_2}(\lambda) < \overset{2}{\beta_2}(\lambda)$. Therefore Lemma 10 applies, with $x = \beta^1(\lambda)$, $y = \beta^2(\lambda)$: there exists a boundary point u of D such that:

$$u \varepsilon \partial_w(S) \text{ and } \overset{2}{\beta_1}(\lambda) < u_1 < \overset{1}{\beta_1}(\lambda), \ \overset{1}{\beta_2}(\lambda) < u_2 < \overset{2}{\beta_2}(\lambda)$$

In particular $u > d(\lambda)$ since $d_1(\lambda) < \overset{2}{\beta_1}(\lambda)$, $d_2(\lambda) < \overset{1}{\beta_2}(\lambda)$.

Define Γ to be the boundary of D. It is a chain because D is *-comprehensive. Moreover we have shown that it satisfies property (31) so the proof is complete.

3. Proof of Lemma 1

For a given vector λ we define

$$U_1(\lambda) = \{z\varepsilon\Lambda / u_1(\lambda - z) \geq u_1(\lambda/2)\}$$

Since u_1 is monotonic the 2 takes all utility level $a_2(\lambda)$ is

$$a_2(\lambda) = \max\{u_2(z_2)/z_2\varepsilon U_1(\lambda)\}$$

We will prove that, under assumption (16), the correspondence $U_1(\cdot)$ itself is monotonic:

$$\lambda < \mu \Rightarrow U_1(\lambda) \subseteq U_1(\mu) \qquad (32)$$

This implies at once that $a_2(\cdot)$ is monotonic.

We suppose that property (32) does not hold and derive a contradiction. For some λ, μ, $\lambda < \mu$, the set $U_1(\lambda)$ is not contained in $U_1(\mu)$. Notice that $U_1(\lambda)$ contains $\lambda/2$ and $\lambda/2$ belongs to $U_1(\mu)$. Therefore $U_1(\lambda)$ must contain a point outside $U_1(\mu)$. As both sets $U_1(\lambda)$, $U_1(\mu)$ are comprehensive and closed, this implies that their boundaries do intersect:

$$\text{for some } z\varepsilon\Lambda \quad u_1(\lambda - z) = u_1(z/2) = \alpha; \text{ and } u_1(\mu - z) = u_1(\mu/2) = \beta \qquad (33)$$

(by strict monotonicity of u_1, the boundary of $U_1(\lambda)$ has equation $u_1(\lambda - z) = u_1(\lambda/2)$). Define $x_0 = \lambda/2$, $y_0 = \lambda - z$, $e = \mu/2 - \lambda/2$. By construction $e\varepsilon\ R^p_+$, $e \neq 0$ so that the two mappings b, c, $b(t) = u_1(x_0 + te)$, $c(t) = u_2(y_0 + te)$ are strictly increasing in t, for $t \geq 0$. By (33) we have

$b(0) = c(0) = \alpha$, $b(1) = c(2) = \beta$

Thus the inverse mappings b^{-1}, c^{-1} map $[\alpha,\beta]$ into $[0,1]$ and $[0,2]$ respectively. Next we compute their derivatives $(b^{-1})'$ and $(c^{-1})'$ at some point γ, $\alpha < \gamma < \beta$:

$$(b^{-1})'(\gamma) = 1/(u_1'(x_0+te)\cdot e) \text{ where } u_1(x_0+te) = \gamma$$

$$(c^{-1})'(\gamma) = 1/(u_1'(y_0+te)\cdot e) \text{ where } u_1(y_0+te) = \gamma$$

In view of assumption (16) we have

$$(u_1'(x_0+te)\cdot e) < 2(u_1'(y_0+te)\cdot e) \Rightarrow (c^{-1})'(\gamma) < 2(b^{-1})'(\gamma)$$

The latter inequality contradicts the earlier observation that b^{-1}, c^{-1} map $[\alpha,\beta]$ into $[0,1]$ and $[0,2]$ respectively, which concludes the proof of Lemma 1.

4. Proof of Theorem 4

By Lemma 2 we need only to show the if statement. We assume (17) and define as follows a selection f of the Pareto correspondence ∂S:

$$\text{for all } \lambda\varepsilon\Lambda: \quad f_1(\lambda) = \sup\,\{\beta(a_2(\mu),\ \lambda\wedge\mu)/\mu\varepsilon\Lambda,\ a_2(\mu)\varepsilon I_2(\lambda\ \ \mu)\}$$
$$f_2(\lambda) = \alpha(f_1(\lambda),\lambda) \tag{34}$$

For a given λ, we can choose $\mu = \lambda$ to ensure $a_2(\lambda)\ \varepsilon\ I_2(\lambda)$, so the supremum defining f makes sense. Next $\beta(a_2(\mu)),\lambda\wedge\mu)$ belongs to $I_1(\lambda\wedge\mu)$. Since S is monotonic w.r.t.λ we have $I_1(\lambda\wedge\mu)\subseteq I_1(\lambda)$. Thus $f_1(\lambda)$ is in $I_1(\lambda)$ and $f_2(\lambda)$ is well defined.

By construction $f(\lambda)$ is Pareto optimal in $S(\lambda)$ (Lemma 3). We show that it is individually rational as well. First take $\mu = \lambda$ in the definition of $f_1(\lambda)$:

$$f_1(\lambda) \geq \beta(a_2(\lambda),\lambda) \geq d_1(\lambda)$$

(the latter inequality follows from $(d_1(\lambda), a_2(\lambda)) \varepsilon S(\lambda)$).

Next by Lemma 3, (u_1, u_2) is an interior point of $S(\lambda)$ if and only if $u_1 < \beta(u_2, \lambda)$, therefore assumption (17) writes:

$$\text{for all } \lambda, \mu, \ \beta(a_2(\mu), \lambda \wedge \mu) < a_1(\lambda) \text{ or } a_2(\mu) \notin I_2(\lambda \wedge \mu)$$

Thus $f_1(\lambda) < a_1(\lambda)$, implying

$$f_2(\lambda) = \alpha(f_1(\lambda),\lambda) > \alpha(a_1(\lambda),\lambda) > d_2(\lambda)$$

We show finally that f is monotonic w.r.t.λ. Surely f_1 is monotonic: when λ rises to $\lambda^1(\lambda^1 > \lambda)$ the set $I_2(\lambda \wedge \mu)$ expands $(I_2(\lambda \wedge \mu) \quad I_2(\lambda^1 \wedge \mu))$ and so does the function that is being maximized:

$$\beta(a_2(\mu),\lambda \wedge \mu) < \beta(a_2(\mu),\lambda^1 \wedge \mu)$$

It remains to show that f_2 is monotonic as well. As α is decreasing in u_1 we have:

$$f_2(\lambda) = \inf\{\alpha(\beta(a_2(\mu),\lambda \wedge \mu),\lambda)/\mu \varepsilon \Lambda, \ a_2(\mu) \varepsilon I_2(\lambda \wedge \mu)\}$$

Thus it is enough to check, for $\lambda < \lambda^1$, the inequality:

$$\alpha(\beta(a_2(\mu), \ \lambda^1 \wedge \mu),\lambda^1) > \alpha(\beta(a_2(\mu),\lambda \wedge \mu),\lambda) \tag{35}$$

Apply the supermodularity assumption (property i) to $\lambda_0 = \lambda$, $\mu_0 = \lambda^1 \wedge \mu$, and $u_2 = a_2(\mu) \varepsilon I_2(\lambda \wedge \mu)$:

$$\alpha(\beta(a_2(\mu),\mu_0),\lambda_0 \vee \mu_0) > \alpha(\beta(a_2(\mu),\lambda_0 \wedge \mu_0),\lambda_0)$$

As $\lambda_0 \vee \mu_0 = \lambda \vee (\lambda^1 \wedge \mu) < \lambda^1$ we get

$$\alpha(\beta(a_2(\mu),\lambda^1 \wedge \mu),\lambda^1) > \alpha(\beta(a_2(\mu),\lambda^1 \wedge \mu), \ \lambda_0 \vee \mu_0)$$

Finally $\lambda_0 \wedge \mu_0 = \lambda \wedge (\lambda^1 \wedge \mu) = \lambda \wedge \mu$ so the last two inequalities together imply (35), and the proof is complete.

Remark 4

The above construction of a monotonic solution of Φ gives asymmetrical roles to the two agents. Of course these roles can be exchanged and the same argument would show that g defined as follows is another monotonic solution:

$$\text{for all } \lambda\varepsilon\Lambda \quad \begin{cases} g_1(\lambda) = \beta(g_2(\lambda),\lambda) \\ g_2(\lambda) = \sup\{\alpha(a_1(\mu),\lambda\wedge\mu)/\mu\varepsilon\Lambda,\ a_1(\mu)\ \varepsilon\ I_1(\lambda\wedge\mu)\} \end{cases} \tag{36}$$

In fact the two solutions f, g are the two extreme points of $M(\Phi)$ in the sense that any monotonic solution h of Φ is no better for agent 2 and no worse for agent 1 than g:

$$f_1(\lambda) \leq h_1(\lambda) \leq g_1(\lambda)$$

$$g_2(\lambda) \leq h_2(\lambda) \leq f_2(\lambda)$$

This follows from the general characterization result Theorem 5 (see below).

5. Proof of Lemma 4

First we assume p = 2 (two goods) and prove property (25). Pick u_1, u_1' as in (25) and define

$$U_1 = \{z_1/u_1(z_1) \geq u_1\} \qquad U_1' = \{z_1/u_1(z_1) \geq u_1'\}$$

Denoting σ the symmetry around w/2, $(\sigma(x) = w-x)$ the implication (25) can be written as:

$$\begin{aligned} &\max\{u_2(z_2)/z_2\varepsilon\sigma(U_1)\} = \max\{u_2(z_2)/z_2\varepsilon\sigma(U_1')+\varepsilon\} \Rightarrow \\ &\max\{u_2(z_2)/z_2\varepsilon\sigma(U_1)+\delta\} \leq \max\{u_2(z_2)/z_2\varepsilon\sigma(U_1')+\varepsilon+\delta\} \end{aligned} \tag{37}$$

Assume for simplicity that the first two maxima are reached at a single commodity vector, namely

$$z_2 = \text{argmax}\{u_2/\sigma(U_1)\}, \quad z_2' = \text{argmax}\{u_2/\sigma(U_1')+\varepsilon\}$$

(this uniqueness assumption could be relaxed). We claim that z_2' is Northwest of z_2:

$$z_{21}' < z_{21}, \; z_{22}' > z_{12} \tag{38}$$

(where z_{21}' is the first coordinate of z_2', and so on). Otherwise z_2' is southeast of $z_2(z_{21}' > z_{21}, \; z_{22}' < z_{12})$ since z_2' and z_2 are on the same indifference curve of u_2. By convexity of the upper contour sets of u_2 this implies that the tangent to $\sigma(U_1)$ at z_2 is steeper than the tangent to $\sigma(U_1') + \varepsilon$ at z_2': see Figure 8.

Note that $\sigma(U_1')$ is a subset of $\sigma(U_1)$ (since $u_1' > u_1$) and that the boundary of $\sigma(U_1)$ must intersect the interval $]z_2'-\varepsilon, \; z_2']$ at some point x (see Figure 8). Because z_2 is Northwest of x, the tangent to $\sigma(U_1)$ at x is not less steep than at z_2 (by the convexity of the preferences u_1). We conclude that the tangent to $\sigma(U_1)$ at x is steeper than the tangent to $\sigma(U_1')$ at $z_2'-\varepsilon$). This contradicts the assumption (26), implying that the slope $(u_1)_1 \; / \; (u_1)_2(x_1,x_2)$ is increasing in the second coordinate x_2.

We have proved (38). The proof of (25) rests now on the following fact: Given z_2, z_2' both on the same indifference curve of u_2, with z_2' Northwest of z_2, denote by Δ the tangent to this indifference curve at z_2. Then the indifference curve of u_2 through $z_2'+\delta$ is on or above the line $\Delta+\delta$, namely the parallel to Δ through $(z_2+\delta)$: see Figure 9. This fact follows from the normality of agent 2's preferences (assumption (26) applied to u_2). We omit the straightforward proof.

Finally, observe that $\sigma(U_1)+\delta$ lies on or below $\Delta+\delta$ (it is tangent at $z_2 + \delta$) whereas $z_2' + \delta$ belongs to $\sigma(U_1') + \varepsilon + \delta$. Hence:

$$\max\{u_2/\sigma(U_1)+\delta\} \leq \max\{u_2/\Delta+\delta\} \leq u_2(z_2'+\delta) \leq \max\{u_2/\sigma(U_1') + \delta + \varepsilon\}$$

as was to be proved.

We have proved Lemma 4 in the case of two goods (p=2). With an arbitrary number of goods, we must prove the implication (37) for any two vectors ε, δ such that:

$$\varepsilon \geq 0,\ \delta \geq 0,\ \text{for all } k = 1,\dots,p:\quad \varepsilon_k \cdot \delta_k = 0$$

The proof is by induction on the number m of coordinate k such that $\varepsilon_k > 0$ or $\delta_k > 0$. We have cleared the case m = 2. If m = 3 and, for instance, $\delta_1 > 0$, $\delta_2 > 0$, $\varepsilon_3 > 0$, one can combine (37) for $\delta_1 > 0$, $\varepsilon_3 > 0$ and for $\delta_2 > 0$, $\varepsilon_3 > 0$. We omit the details.

6. Characterization of the existence of a monotonic surplus-sharing in two person problems

Given is a two person problem $\Phi = (d,S)$. Our result and its proof use extensively the two representations α, β of the feasible set correspondence.

$$\text{for all } \lambda\varepsilon\Lambda, \text{ all } u\varepsilon[0,L]^2 : u\varepsilon S(\lambda) \Leftrightarrow \{u_1\varepsilon I_1(\lambda) \text{ and } u_2 \leq \alpha(u_1,\lambda)\}$$
$$\Leftrightarrow \{u_2\varepsilon I_2(\lambda) \text{ and } u_1 \leq \beta(u_2,\lambda)\}$$

Recall also the notations

$$a_1(\lambda) = \beta(d_2(\lambda),\lambda),\ a_2(\lambda) = \alpha(d_1(\lambda),\lambda) \text{ for all } \lambda\varepsilon\Lambda$$

We can transform every mapping h from Λ into $[0,L]^2$ into a mapping $\Theta(h)$ from

Λ into $[0,L]^2$, defined as follows:

$$\text{for all } \lambda\varepsilon\Lambda \quad \Theta_1(h_1)(\lambda) = \sup_{\mu\varepsilon\Lambda}\{\beta(\alpha(h_1(\mu),\mu),\lambda\wedge\mu)\} \tag{39}$$

$$\Theta_2(h_2)(\lambda) = \sup_{\mu\varepsilon\Lambda}\{\alpha(\beta(h_2(\mu),\mu),\lambda\wedge\mu)\}$$

and $\Theta(h)(\lambda) = (\Theta_1(h_1)(\lambda),\Theta_2(h_2)(\lambda))$

Notice that $\Theta(h)$ is well defined only if $h_i(\lambda)$ is in $I_i(\lambda)$ for all $i = 1, 2$ and all λ. This holds true in particular if $h(\lambda)\varepsilon S(\lambda)$ for all λ.

By taking $\mu = \lambda$ in (39) we get

$$\Theta_1(h_1)(\lambda) > \beta(\alpha(h_1(\lambda),\lambda) > h_1(\lambda)$$

whence $\Theta(h) > h$.

<u>Theorem 5</u> Suppose $\partial S(\lambda) = \partial_w S(\lambda)$ for all λ.

Given a monotonic two person surplus-sharing problem $\Phi = (d,S)$, consider the sequence $d^o = d$, $d^1 = \Theta(d^o)$, $d^{t+1} = \Theta(d^t)$, all $t = 1,2,\ldots$.

There exists a monotonic solution of $\Phi(M(\Phi) \neq \phi)$ if and only if all mappings d^t are feasible:

for all $t = 1,2,\ldots$, for all λ, $d^t(\lambda)\varepsilon S(\lambda)$

<u>Proof</u>

Notice that $d^1 = \Theta(d)$ is defined as follows:

$$d_1^1(\lambda) = \sup_{\mu} \beta(a_2(\mu),\lambda\wedge\mu), \quad d_2^1(\lambda) = \sup_{\mu} \alpha(a_1(\mu),\lambda\wedge\mu)$$

We prove <u>only if</u>. Suppose f is a monotonic solution of Φ. Then for every λ, μ in Λ we have:

$$f_2(\lambda\wedge\mu) < f_2(\mu) < a_2(\mu) \Rightarrow \beta(a_2(\mu),\lambda\wedge\mu) < \beta(f_2(\lambda\wedge\mu),\lambda\wedge\mu) < f_1(\lambda\wedge\mu) < f_1(\lambda)$$

Taking the supremum w.r.t. μ we obtain $d_1^1(\lambda) \leq f_1(\lambda)$. The proof of $d_2^1 \leq f_2$ is similar. Thus $d^1 \leq f$, implying that d^1 is feasible since f_1 is.

Observe next that d^1 is a monotonic mapping of λ since $\beta(.,\lambda)$ and $\alpha(.,\lambda)$ are monotonic in λ, too. We have shown that (d^1,S) is a monotonic surplus-sharing problem. Repeating the argument we conclude that (d^t,S) is a monotonic surplus-sharing problem for all $t = 1,2,\ldots$ so that $d^t(\lambda)$ is feasible for all λ and all t.

We prove <u>if</u>. Since $\Theta(h) \geq h$, the sequence d^t is nondecreasing

$$d \leq d^1 \leq \ldots \leq d^t \leq d^{t+1} \leq \ldots$$

Denote by d^* the limit of this sequence:

$$d^*(\lambda) = \sup_{t=1,2,\ldots} d^t(\lambda)$$

As $S(\lambda)$ is closed, it must contain $d^*(\lambda)$. By the argument of the "only if" proof, every monotonic solution of Φ is bounded below by d^* (we showed $f \geq d \Rightarrow f \geq d^1$ and this argument can be repeated). We check now that a take-all solution above d^* is one desired monotonic solution.
Consider the "2 takes-all" solution:

$$f_1(\lambda) = d_1^*(\lambda) \qquad f_2(\lambda) = \alpha(d_1^*(\lambda),\lambda)$$

Clearly $f(\lambda)$ belongs to $\partial_w S(\lambda)$ and $f \geq d^*$, implying $f \geq d$. Moreover f_1 is monotonic w.r.t. λ as each d^t is. It remains only to show that f_2 is monotonic w.r.t. λ as well.

Fix λ, λ' in Λ with $\lambda \leq \lambda'$. Then for all $t = 1,2,\ldots$ we have:

$$\beta(\alpha(d_1^t(\lambda'),\lambda'),\lambda) \leq \sup_{\mu} \beta(\alpha(d_1^t(\mu),\mu),\lambda \wedge \mu) = d_1^{t+1}(\lambda)$$
$$\Rightarrow \alpha(d_1^{t+1}(\lambda),\lambda) \leq \alpha(\beta(\alpha(d_1^t(\lambda'),\lambda'),\lambda),\lambda)$$

By Lemma 3, the latter inequality is just

$$\alpha(d_1^{t+1}(\lambda),\lambda) < \alpha(d_1^{t}(\lambda'),\lambda')$$

Moreover, α is continuous on $I_1(\lambda)$, $I_1(\lambda')$ so that we may take the limits on both sides of the inequality when t goes to infinity, yielding the desired monotonicity of f_2.

Remark 5

The proof of Theorem 5 actually exhibits the most advantageous element of $M(\Phi)$ for agent 2 (and a symmetrical construction gives the most advantageous element for agent 1).

Another side product of the above proof is a new insight into that of Theorem 4. When the correspondence S is super modular, one shows easily that the transformation Θ satisfies

$$\text{for all selection } d \text{ of } S,\ \Theta^2(d) = \Theta(d)$$

Hence, the existence of a monotonic solution to (d,S) is equivalent to the feasibility of $\Theta(d)$:

$$\text{for all } \lambda\varepsilon\Lambda \quad \Theta(d)(\lambda)\varepsilon S(\lambda)$$

Moreover the most advantageous such solution for agent 2 is

$$f(\lambda) = (\Theta_1(d_1)(\lambda),\ \ \alpha(\Theta_1(d_1)(\lambda),\lambda))$$

and the most advantageous for agent 1 is

$$g(\lambda) = (\beta(\Theta_2(d_2)(\lambda),\lambda),\ \Theta_2(d_2)(\lambda))$$

This is the result announced in Remark 4 (at the end of the proof of Theorem 4).

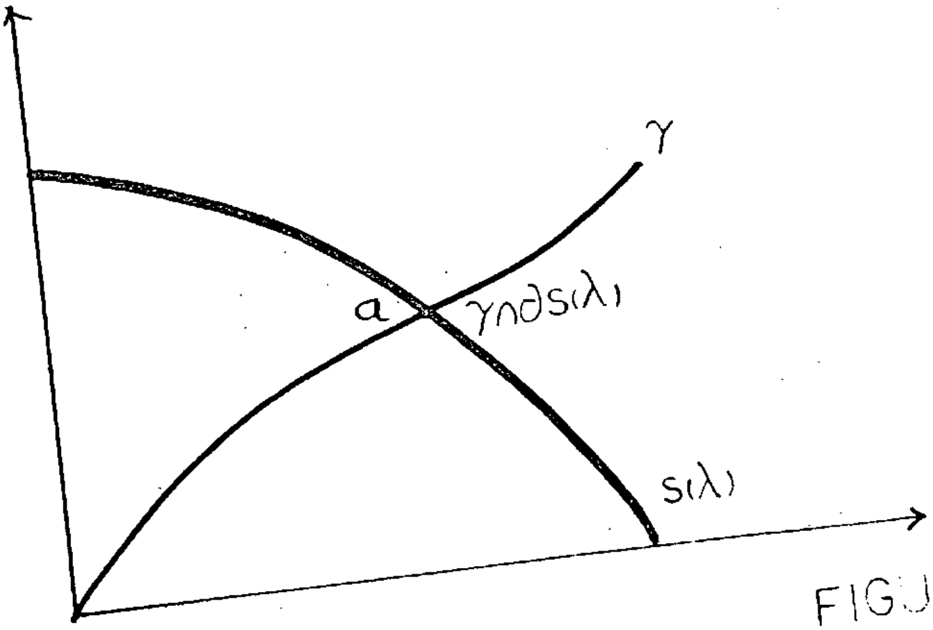


FIGURE 1.A

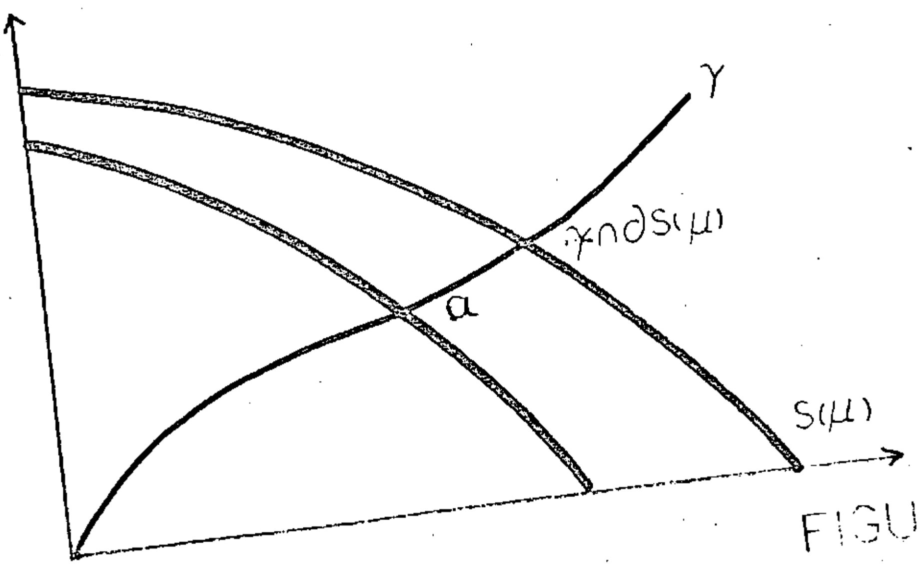


FIGURE 1.B

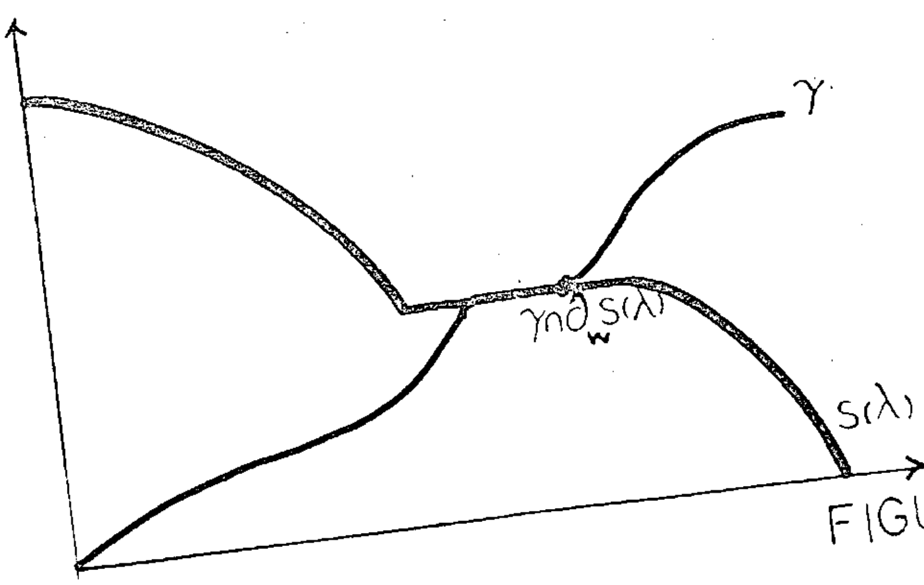


FIGURE 1.C

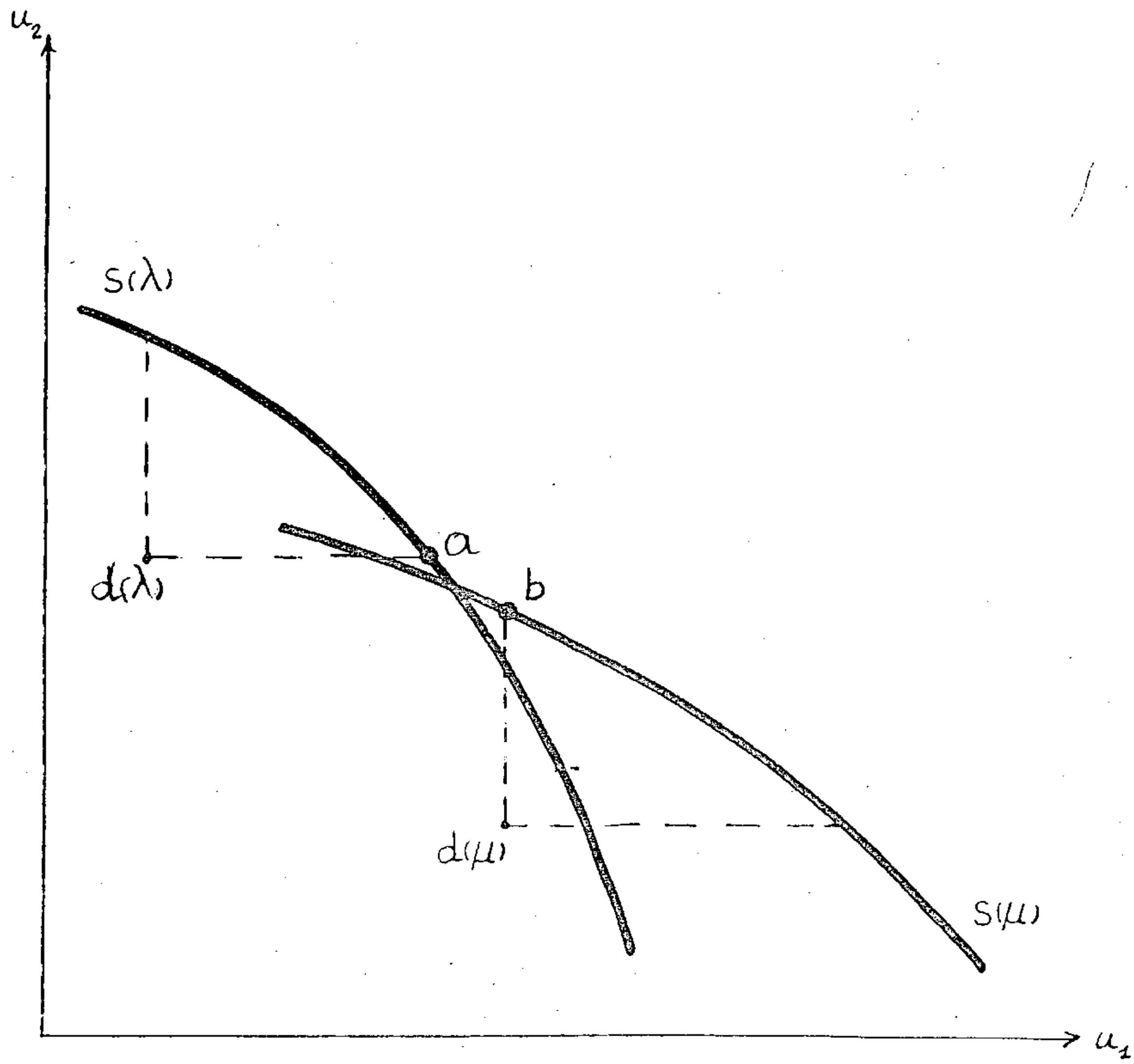


FIGURE 2

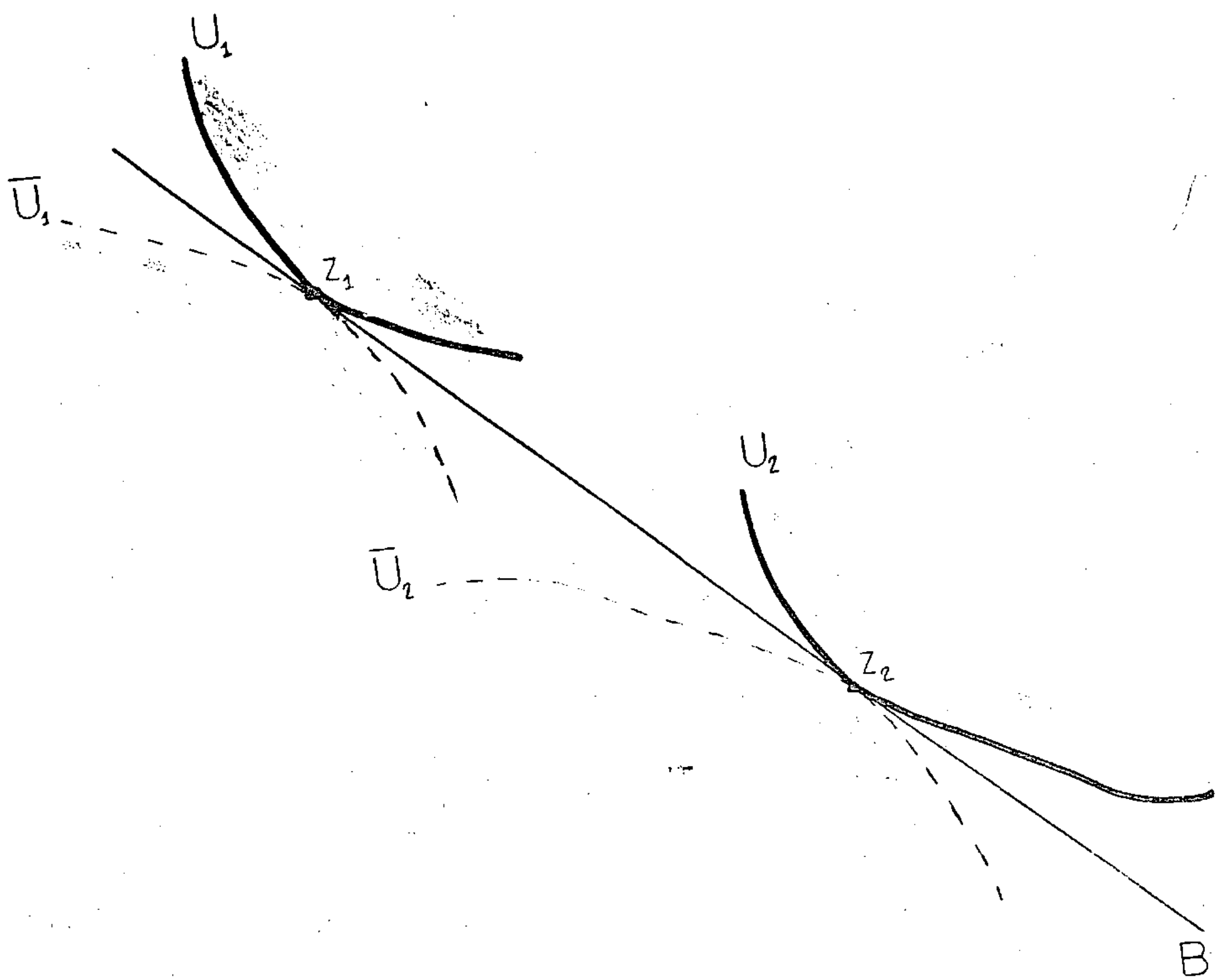
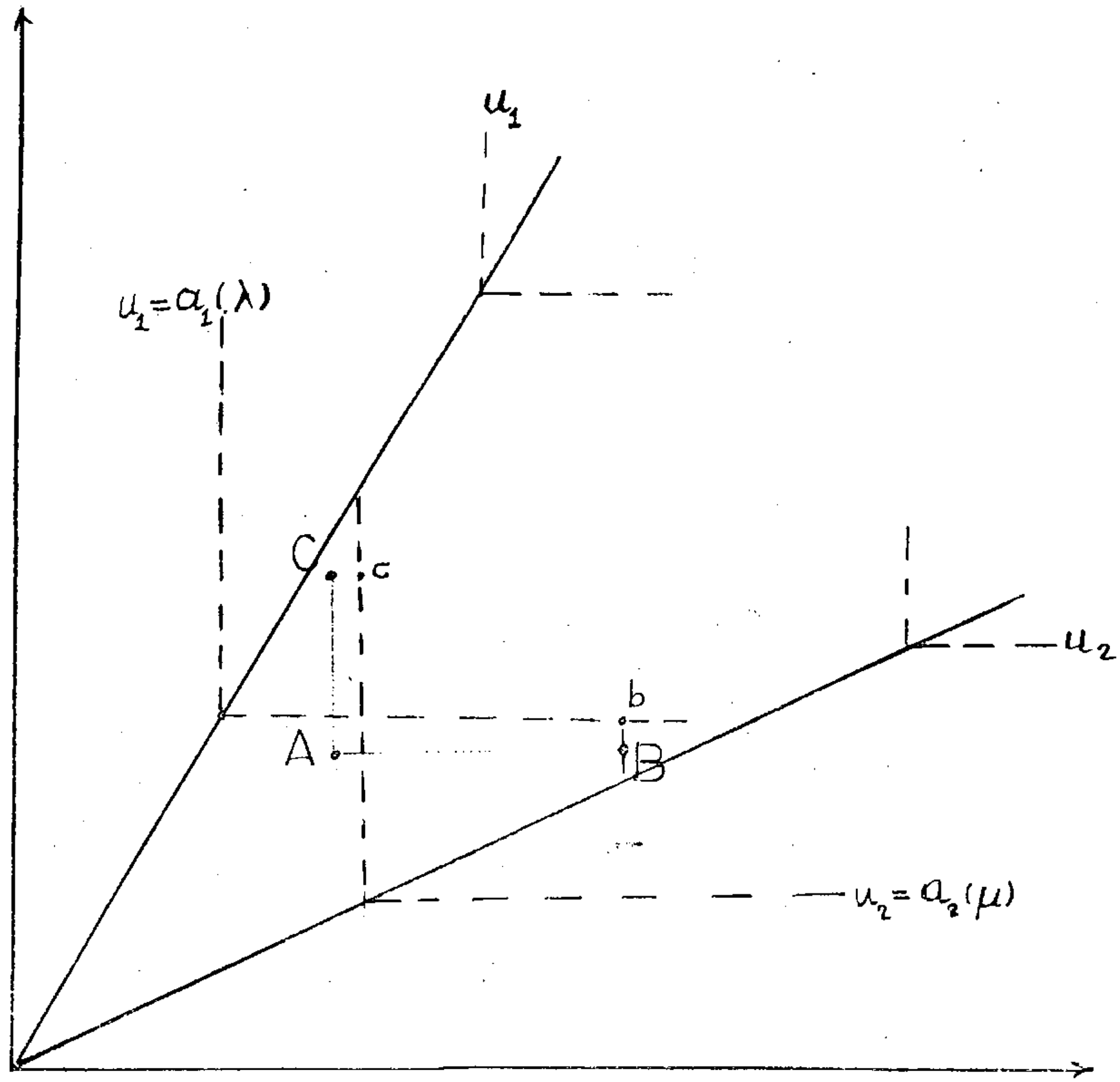


FIGURE 3



$$\begin{cases} A = \frac{1}{2}(\lambda \wedge \mu) \\ B = \frac{1}{2}\lambda \\ C = \frac{1}{2}\mu \end{cases}$$

FIGURE 4

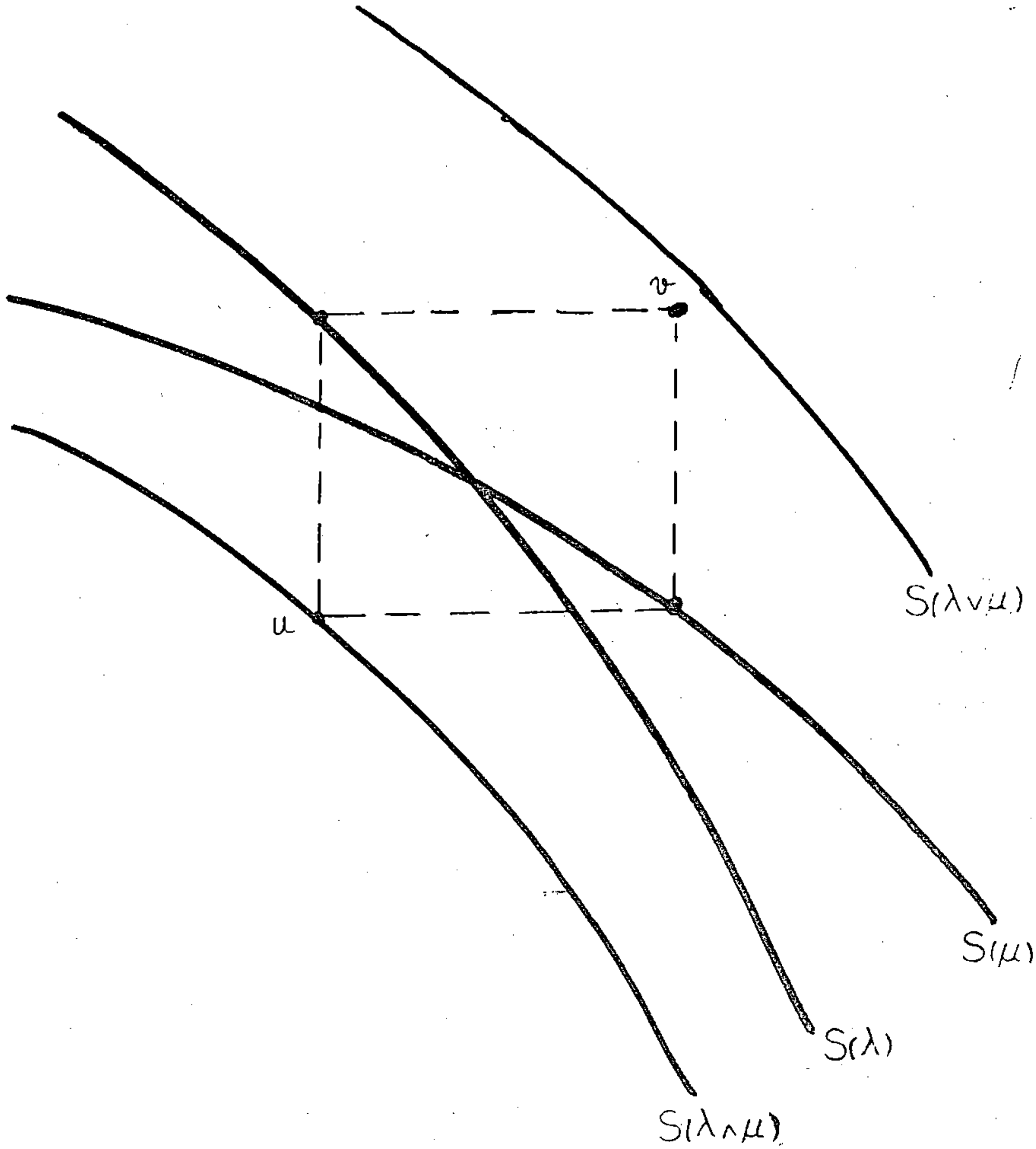


FIGURE 5

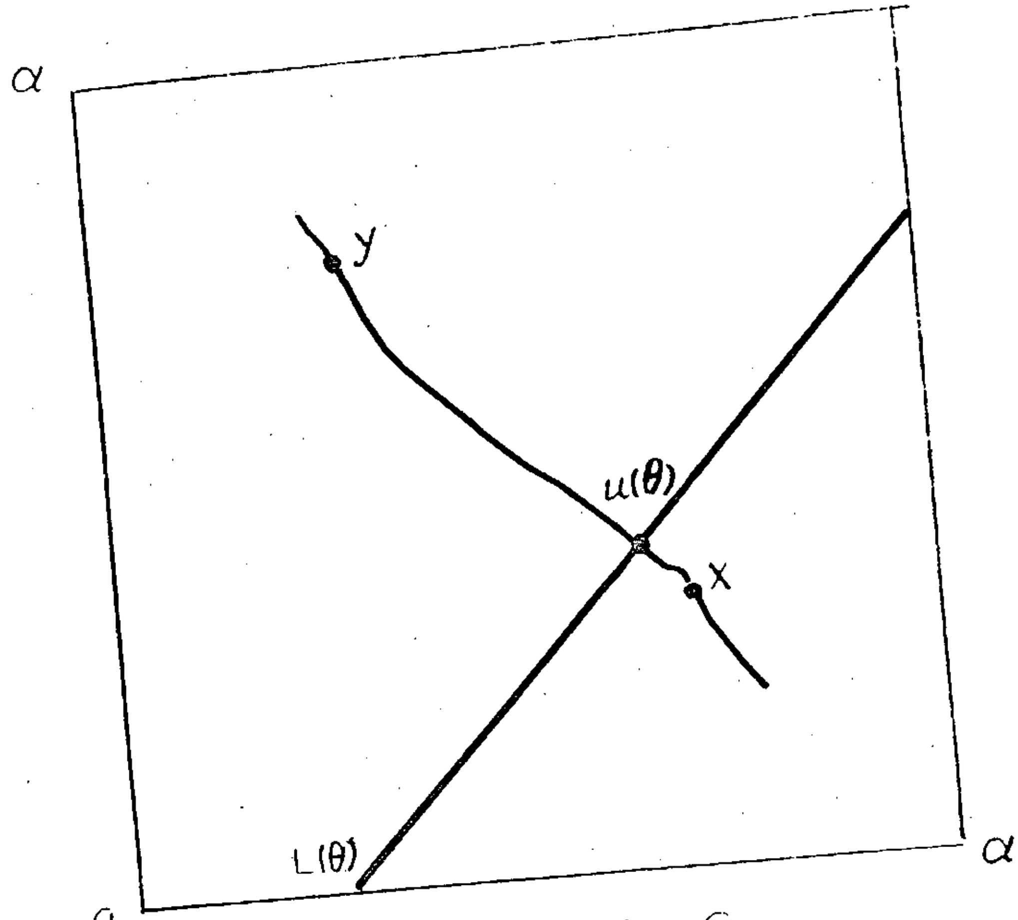


FIGURE 6

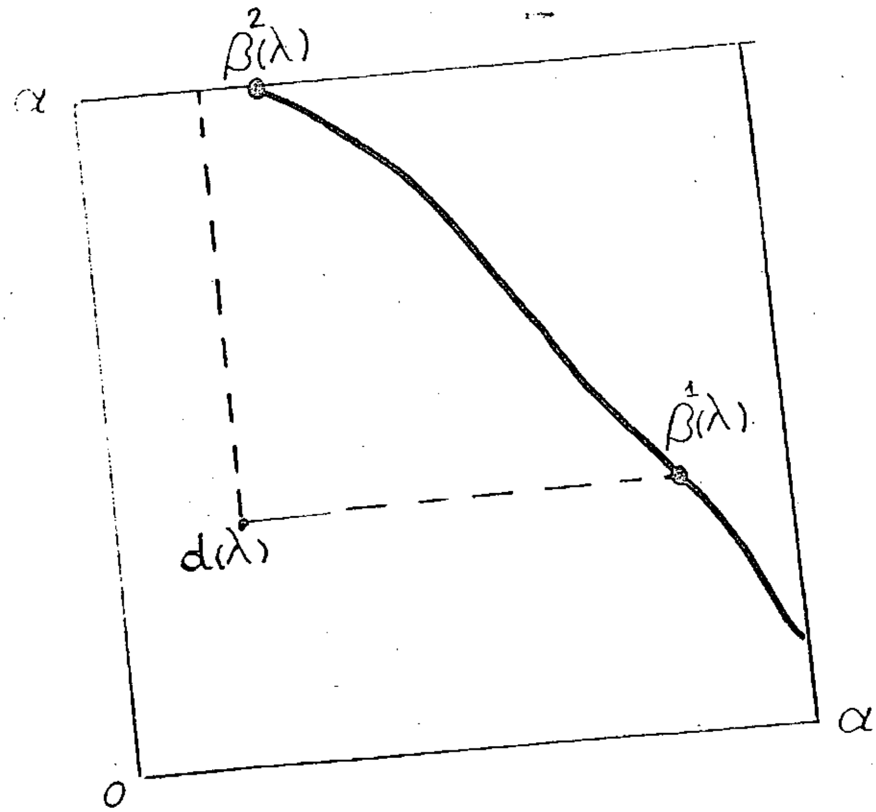


FIGURE 7

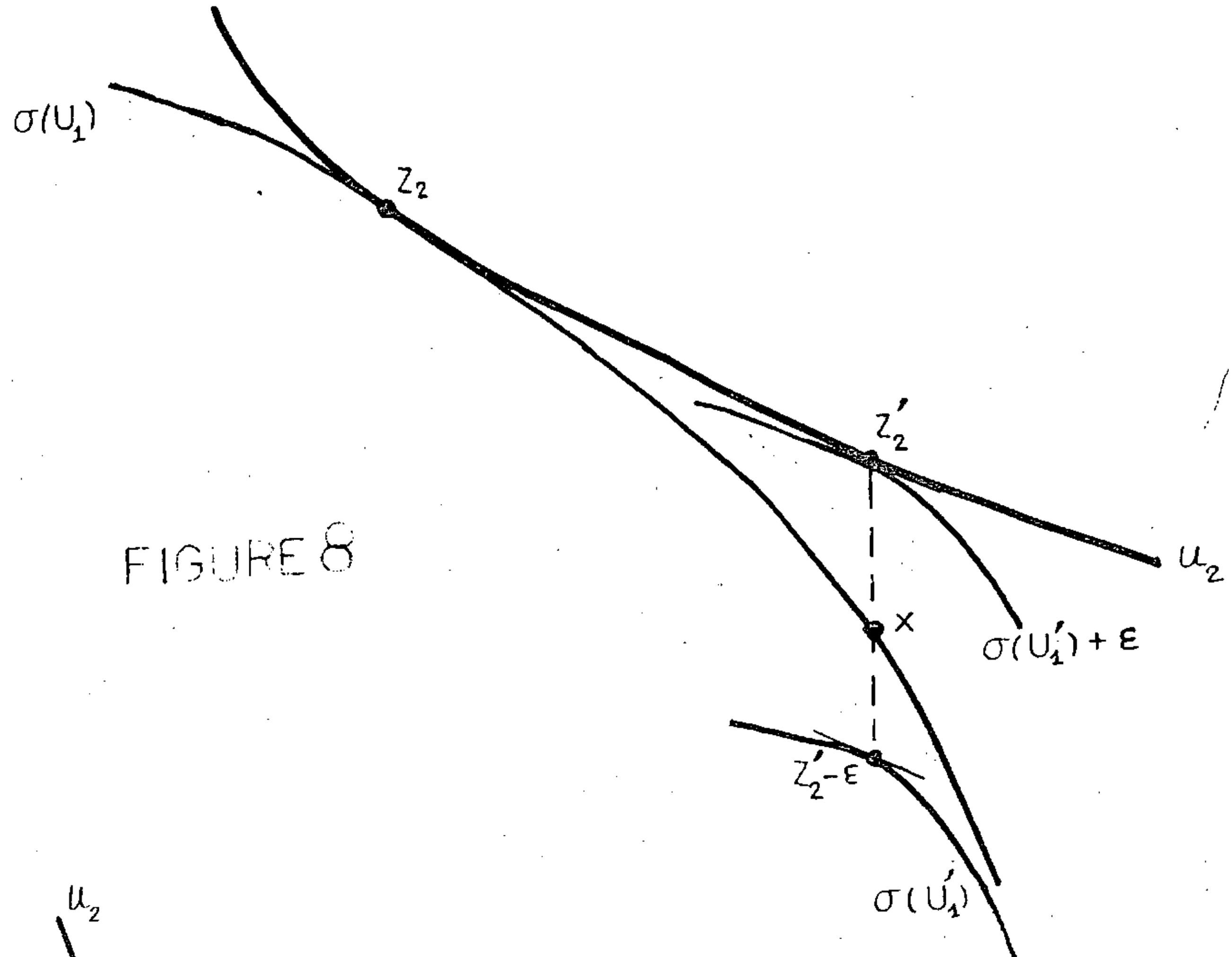


FIGURE 8

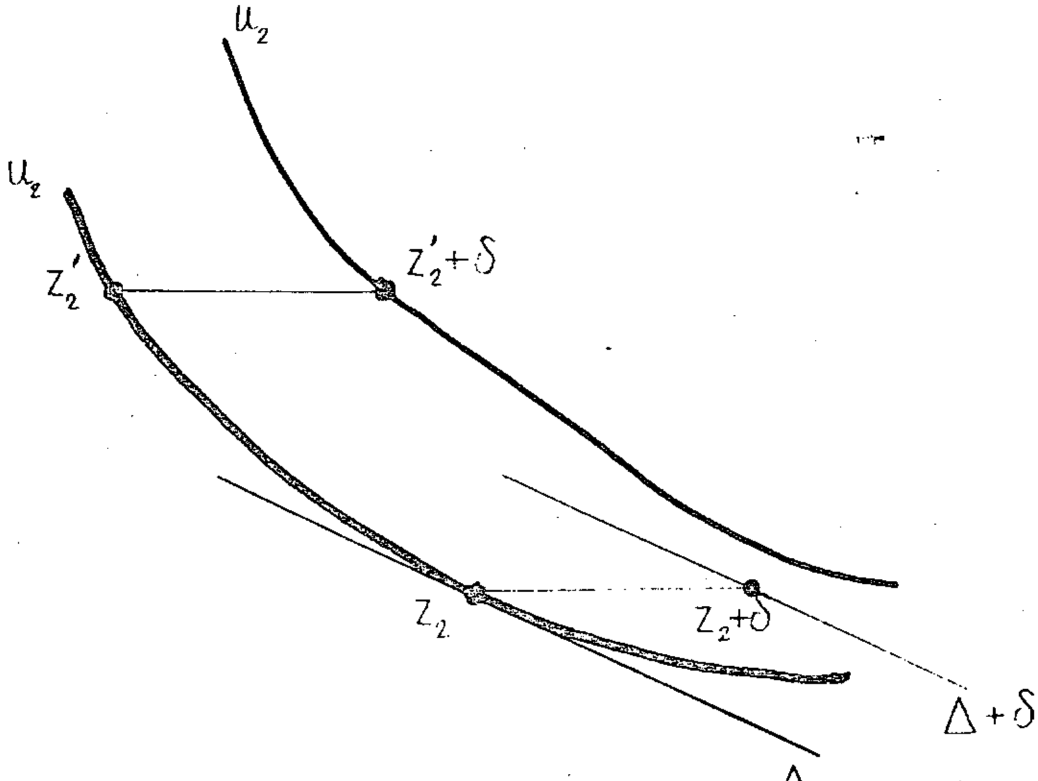


FIGURE 9