

Towards A Theoretical Model of Urban Growth

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Abstract

The overwhelming trend is for urban areas to grow. The challenge is to accentuate the positive impacts of this growth (innovation, art, wealth, etc) while mitigating the negative burdens (crime, pollution, poverty, loss of biodiversity, etc). However, this challenge is complicated by the interconnected physical, biological and social issues. There are several studies on just one aspect of this problem, but we propose a more encompassing approach by looking at the interplay between institutions and ecological processes (topography, economics, etc) using both computational and analytical (mathematical equations) approaches. We used simplified equations to build an intuition towards a more comprehensive modeling framework.

1 Introduction

Cities grow. Not only are cities growing in terms of population size, but the proportion of people living in cities is also growing. One of our biggest challenges in moving forward as a society is to develop sustainable policies and practices, but we have no hope of doing

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so unless we understand how and why our cities are growing. The aim of this paper is to present a stylized model of urban growth using mathematical equations to incorporate economic and non-economic factors. This model is the first step in a framework that seeks to conciliate well-understood and readily measureable economic factors of urban growth with more elusive, but equally fundamental, physical and social environment factors using a system of differential equations. As traditional cities give way to major metropolitan areas or the megalopolis, it is clear that traditional models of urban growth are inadequate to describe the role of culture, sense of place, weather and the dating market have played in urban growth.

The rest of this paper is organized as follows: in the next section we briefly describe the recent trends in urban growth and some traditional approaches to understanding this growth. Then, we introduce our stylized model of urban growth followed by some analytical and numerical results. Then we conclude with some implications of the model presented.

2 Background

In this section, first we will describe how cities are growing and motivate why economic and non-economic factors are vital components of any model of urban growth. Then, we will discuss some traditional approaches that have been used in various disciplines that motivates our modeling approach.

2.1 Growth and Change

The proportion of Americans living in urban areas has increased steadily since the 1900's (Pew Research Center, 2009) and now approaches 80%. Urban growth or decline results from one of three factors: natural increase/decrease from births and deaths, net migration, and reclassification of rural areas as urban. In developing countries natural increase dominates urban growth (UN State of the World Report, 2007). In contrast, domestic and foreign migration dominates urban growth in the US (US Census Special Report, 2000). For example, between 1995 to 2000, over 45% of US residents moved at least once and certain groups (such as young highly educated and retirees) tend to move to distinctly different types of cities (Chen and Rosenthal, 2008). Demographic differences not only impact where people move, but also where they would like to move (Pew Research Center, 2009). Although the US population has become less mobile, the Pew Research Center (2009) finds that nearly half of the public would rather live somewhere else.

Although the reasons why movers move, and stayers stay varies somewhat among different demographic and geographic groups, in a recent survey (Pew Research Center, 2009) the most common reason cited for moving was for a job or business, cited by 44%. However, this economic factor was followed by several non-economic factors for moving: good place to raise children (36%), family ties (35%), education or schooling (29%). The next economic factor is only cited by 24% of the movers as a major reason why they moved. Although economics appears to drive migration, three of the top five reasons for moving are not economic and the difference between the top economic and non-economic reasons

is only 8 percentage points. There is a much greater distinction if we look at reasons why stayers stay. Family ties is mentioned by 74% of stayers as a major reason why they stayed. The next four reasons are also non-economic and the top economic reason, jobs or business, was only mentioned by 40% of the stayers for a difference of 34 percentage points between the top economic and non-economic reason why stayers stay.

Deciding not to move is just as big a decision as deciding to move and clearly non-economic reasons play an integral role in this calculus. Thus, any model of urban growth must be flexible enough to consider at least some of these non-economic factors. Next we will review some past approaches to studying urban growth.

2.2 Traditional Approaches

In a recent New York Times Blog (Judson, 2009), Steven Strogatz mentions mathematical modeling and its application to studying cities:

One of the pleasures of looking at the world through mathematical eyes is that you can see certain patterns that would otherwise be hidden.

The power of mathematics is in abstraction and looking for patterns or trends. If these patterns exist, then we may be able to translate new problems into a framework similar to that of existing problems. Doing so allows us to leverage all the tools and methods that have been developed to solve similar problems. One example of this is considering cities as distinct units that compete with each other for resources (Dendrinos and Mullally, 1985) in a predator-prey dynamic. The economics literature has taken advantage of these types of

mathematical formulations and have used them to, among many other things, describe the growth of cities (see Cordoba, 2008 for a review of some recent success and limitations). One pattern that has proven to be ubiquitous among cities in different regions is Zipf's Law or the Rank Size Rule. The rule essentially states that there is a correlation between the population size of a city and its rank (determined by ordering all the cities in a region according to its population size). What is surprising is that this relationship is very robust and similar relationships hold for not just the rank, but also the creativity, wealth, crime and number of gas stations in cities (Bettencourte, et al, 2007).

One critique of these models is that they tend to be too simplistic in their descriptions and restrictive in their assumptions. To address this critique, experimental economists tend to look at large data sets with many variables to describe the problem of interest, in much the same way urban geographers may study urban growth (Knox, 1994). The rise of GIS has revolutionized these fields by allowing the incorporation of spatial data. Among other things, this has helped reveal the importance of place and uncover geographic correlations. However, these models have limitations as well since correlations do not imply causation and confounding factors can lead to misleading results.

There has been a lot of effort in the economics literature to include new factors such as natural amenity (the quality of the physical environment) and it is our goal to build on this foundation by creating the simplest model possible that combines traditional economic and non-economic factors of urban growth. This simple model forms a skeleton upon which more details, and hence more complexity, can be added later to flesh out a more

comprehensive model of urban growth.

3 Mathematical Model

In our stylized model, a city is described using just two factors, population size (N) and infrastructure (K). We assume growth of the city is due solely to net migration, not demographic effects. Many classical economic models assume the population size and infrastructure are either fixed or always at equilibrium. Here we explicitly describe the time rate of change of these variables:

$$\dot{N} = f(N, K) \tag{1}$$

$$\dot{K} = g(N, K), \tag{2}$$

Thus $f(N, K)$ describes the migration process and $g(N, K)$ describes how infrastructure changes over time. In order to have migration, there must be somewhere people migrate to/from; we will call this the hinterland. We assume the hinterland is much bigger than the city and this allows us to assume a constant number of people that can migrate into the city, γ . Then we can assume there is a constant rate of migration out of the city which we normalize to be 1 per capita in the city. However, not everyone that may want to migrate into or out of the city does since there are transaction costs associated with moving. Let $I(N, K, U)$ be the per capita immigration rate into the city and $E(N, K, U)$ is the per capita emigration rate out of the city. Note that these rates

are functions of N and K , and thus change over time. The utility, U , will be described below. Then $\gamma \cdot I$ represents the total rate of actual immigration into the city. Similarly, the total rate of emigration is $N \cdot E$. Then we can write down the differential equation for the rate of change of the population of the city.

$$\dot{N} = \gamma I(N, K, U) - N \cdot E(N, K, U), \quad (3)$$

Next we assume infrastructure (K) to be a proxy for the physical, social, and cultural capital contained within a city. Capital may decay over time at some rate, δ . This is not only true of physical capital, but social networks also require energy to maintain. Capital grows with investment. We assume an average rate of investment (s) based on the level of productivity ($Y(N, K)$) in the city, we can think of this as an income tax. According to classical economic theory, the productivity of the city should, at minimum, be a function of the labor supply and infrastructure. To preserve the constant returns to scale principle, we assume a Cobb-Douglas formulation of the production function ($Y(N, K)$) where the exponents sum to one. Then we can explicitly write down the production function and the rate of change of infrastructure

$$Y(N, K) = N^{\alpha_1} K^{\beta_1} \quad (4)$$

$$\text{where } \alpha_1 + \beta_1 = 1 \quad (5)$$

$$\dot{K} = s \cdot Y(N, K) - \delta K. \quad (6)$$

where α_1 is the relative strength of labor pool to productivity and β_1 is the relative strength of infrastructure to productivity.

Now we need to explain why people move into the city. As we stated before, economic opportunities, jobs and businesses, are an important reason why people move or stay. However, non-economic factors are also vital. For simplicity, we will split these factors into two categories which we describe as human and natural amenities to put it in the language of economics. Human amenity, $A_h(N, K)$, are the facets of the city, built by people, that add to the enjoyability of that city. For example, walkable streets, a vibrant night life or medical facilities. Natural amenity, $A_n(N, K)$, are natural resources that add to the enjoyability of a city, such as local forests, lakes or micro-climate.

Both of human and natural amenity should be functions of both population size and infrastructure. However, the direction of this relationship is not clear. Consider a lake. Increased infrastructure can represent a dock and services that increase the ability of people to enjoy the lake. It can also represent investment by businesses that may pollute the lake and reduce its amenity value. As the population increases, more people may visit the lake, thus increasing the popularity and reputation of the lake. However, if the increased popularity leads to increased littering or overcrowding, then this can detract from the amenity of the lake. Human amenity operates in an analogous manner. The modeling process is inherently iterative and as a first approximation, we assume a simple linear relationship between amenity and population and infrastructure.

But first we must introduce the utility function ($U(N, K)$), or the measure of how

happy people are. We assume people expect some average level of utility outside the city (\bar{U}). For people in the city, we characterize their utility as being functions of human amenity, natural amenity, and production $Y(N, K)$ (which serves as a proxy for the level of income they might expect to have in the city). We also assume this to have a Cobb-Douglas formulation, but with decreasing returns to scale (all the exponents sum to less than one) since handling time may diminish a person's ability to enjoy a larger provision of goods and services. Then

$$A_h(N, k) = a_{hk}K + a_{hn}N + a_{h0} \quad (7)$$

$$A_n(N, K) = a_{nk}K + a_{nn}N + a_{n0} \quad (8)$$

$$I(N, K, U) = I_0 \left(\frac{\pi}{2} + \arctan(U(N, K) - \bar{U}) \right) \quad (9)$$

$$E(N, K, U) = E_0 \left(\frac{\pi}{2} + \arctan(\bar{U} - U(N, K)) \right) \quad (10)$$

$$U(N, K) = A_h(N, K)^{\alpha_h} A_n(N, K)^{\alpha_n} Y(N, K)^{\alpha_y} \quad (11)$$

$$\text{where } \alpha_h + \alpha_n + \alpha_y < 1 \quad (12)$$

where a_{hk} determines how infrastructure changes human amenity, a_{hn} determines how the population size changes human amenity and a_{h0} is the basic level of human amenity. Similarly, a_{nk} determines how infrastructure changes natural amenity, a_{nn} determines how the population size changes natural amenity and a_{n0} is the basic level of natural amenity. Then I_0 is the maximum per capita immigration rate, and E_0 is the maximum per capita emigration rate. Also, α_h is the relative strength of human amenity to utility,

α_n is the relative strength of natural amenity to utility, α_y and is the relative strength of productivity to utility.

Then we can write our complete set of differential equation to describe urban growth

$$\dot{N} = \gamma I - N(N, K, U) \cdot E(N, K, U) \quad (13)$$

$$\dot{K} = s \cdot N^{\alpha_1} K^{\beta_1} - \delta K \quad (14)$$

$$I(N, K, U) = I_0 \left(\frac{\pi}{2} + \arctan(U(N, K) - \bar{U}) \right)$$

$$E(N, K, U) = E_0 \left(\frac{\pi}{2} + \arctan(\bar{U} - U(N, K)) \right)$$

$$U(N, K) = A_h(N, K)^{\alpha_h} A_n(N, K)^{\alpha_n} Y(N, K)^{\alpha_y}.$$

Next we will describe the analytical results of our model.

3.1 Trivial Equilibrium

Our system of differential equations are (13) and (14). Clearly $K = 0$ is an equilibrium solution of equation (14). Then we can solve for the equilibrium value of population when $K = 0$ (N^0)

$$N^0 = \frac{\gamma I_0 (\pi - 2 \arctan(\bar{U}))}{E_0 (\pi + 2 \arctan(\bar{U}))} \quad (15)$$

this represents a city with no significant infrastructural development. Even though this means that there is no utility gained by residents in the city, there is still an equilibrium

population in the city (N^0). This may seem counterintuitive until we consider the transaction costs associated with moving. Even if you are not happy in the city, there is a cost to move out. Thus N^0 represents a transient population. Although each individual may not stay in the city for very long, the population level remains constant since there are equal numbers of people moving in and moving out. Note, a linear stability analysis is not possible because it would lead to division by zero. Next we look for non-trivial equilibria.

3.2 Non-Trivial Equilibria

Since $\alpha_1 + \beta_1 = 1$ we can rewrite equation 14

$$\dot{K} = sN^{\alpha_1}K^{1-\alpha_1} - \delta K, \quad (16)$$

which we can now set to zero and solve for a non-trivial equilibrium value for the population (N^*)

$$0 = sN^{\alpha_1}K^{1-\alpha_1} - \delta K \quad (17)$$

$$\delta K = sN^{\alpha_1}K^{1-\alpha_1} \quad (18)$$

$$N^{\alpha_1} = \frac{\delta K}{sK^{1-\alpha_1}} \quad (19)$$

$$N^{\alpha_1} = \frac{\delta}{s} \frac{K}{K^{1-\alpha_1}} \quad (20)$$

$$\frac{\delta}{s} K^{\alpha_1} = N^{\alpha_1} \quad (21)$$

$$N^* = \left(\frac{\delta}{s}\right)^{(1/\alpha_1)} K, \quad (22)$$

Then we can solve equation 13 for N .

$$N^* = \gamma \frac{I(N, K, U)}{E(N, K, U)} \quad (23)$$

However, both $I(N, K, U)$ and $E(N, K, U)$ are functions of N and equation (23) is not tractable. However, if we know there is an equilibrium value for K , then we can calculate the equilibrium value of N , and vice versa. Because we can not explicitly solve for the non-trivial equilibria, we will rely on numerical solutions to gain intuition on the qualitative behavior of our model.

3.3 Bi-Stability

With certain parameters we can show that there are two positive stable states and by changing the initial conditions, we can enter different basins of attraction. The existence of two different basins of attraction are significant because then it implies it may be possible to move from one basin to another. The initial conditions determine which basin of attraction the city is in, but if the city receives an influx in either population or infrastructure (perhaps due to some policy change) then it may be possible for a city to change it's long term behavior in this model.

4 Conclusion

We have created a stylized model of urban growth that indicates cities may follow different trajectories based on their starting conditions. Clearly the parameters for each city will

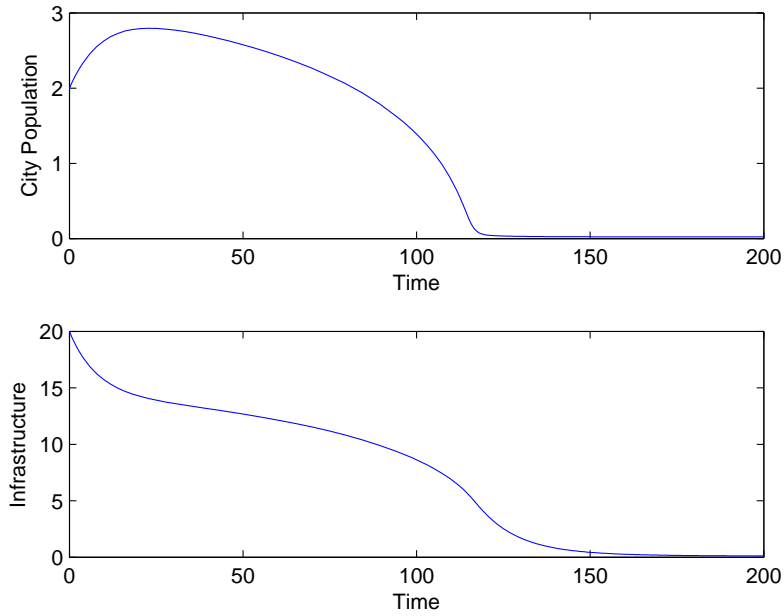


Figure 1: With low initial conditions we enter a basin of attraction of a smaller, but positive non-trivial equilibrium. Parameters: $I_0 = 0.6, E_0 = 0.5, \delta = 0.1, \alpha_h = .3, \alpha_n = .3, \alpha_y = .3, \alpha_1 = .6, \beta_1 = .4, \gamma = .1381, \bar{U} = 3, a_{nk} = 0.5, a_{nn} = 0.5, a_{n0} = 1, a_{hk} = 1.5, a_{hn} = 1.5, a_{h0} = 1$ and $s = 0.25$.

be different and this may account for differences between the trajectory of cities. However, even different cities with the same basic parameters, if one starts with a comparative advantage, it may have qualitatively different behavior. This is the first step in establishing a modeling framework that incorporates both traditional economic and non-economic factors using a system of differential equations.

Under the assumptions of this model, the long run population size of a city is not determined solely by the parameters, but also by the initial conditions. Thus, in order to change the long term population size and infrastructure level, a city can either alter its

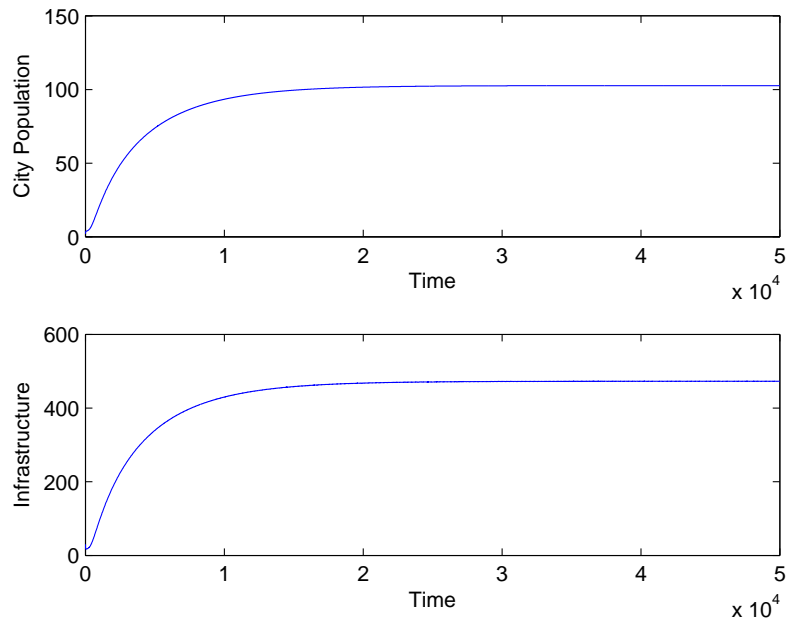


Figure 2: With high initial conditions the city grows rapidly to a larger non-trivial equilibrium.

parameters (for example by making the city more welcoming to immigrants) or change its current state (perhaps through an influx of infrastructure due to capital investment). At the same time, a major disaster that greatly reduces the level of infrastructure or population can cause a city to spiral into a basin of attraction with less desirable long run dynamics.

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